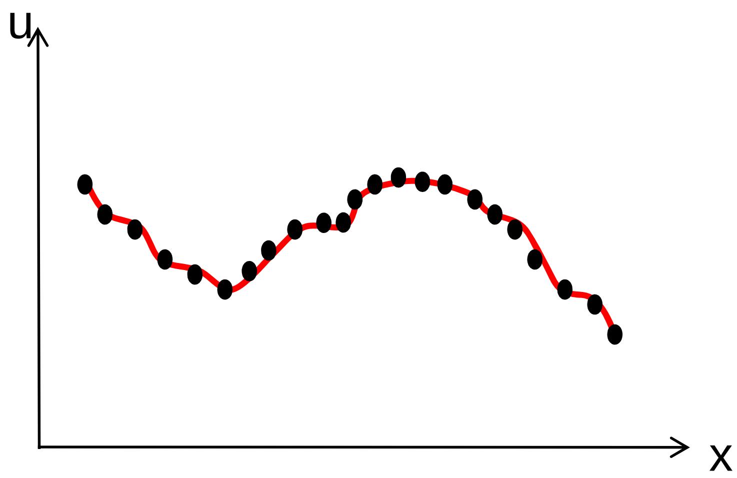
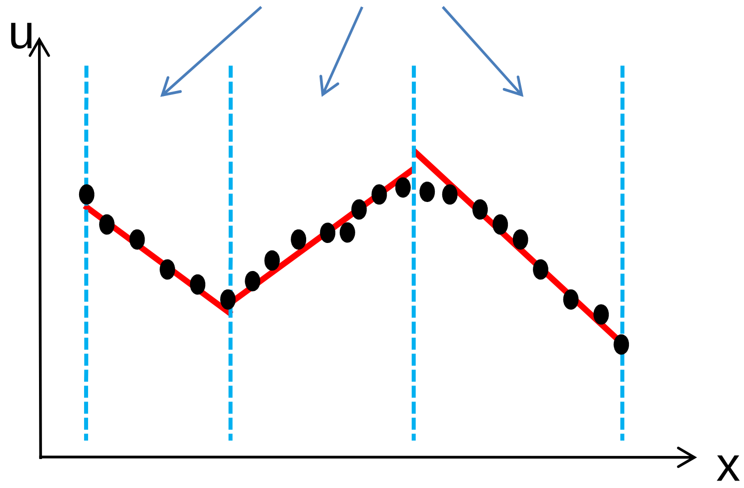
# Interpolation

Consider modelling 1D data (e.g. temperature profile along a 1D rod) …

High order polynomial across whole domain



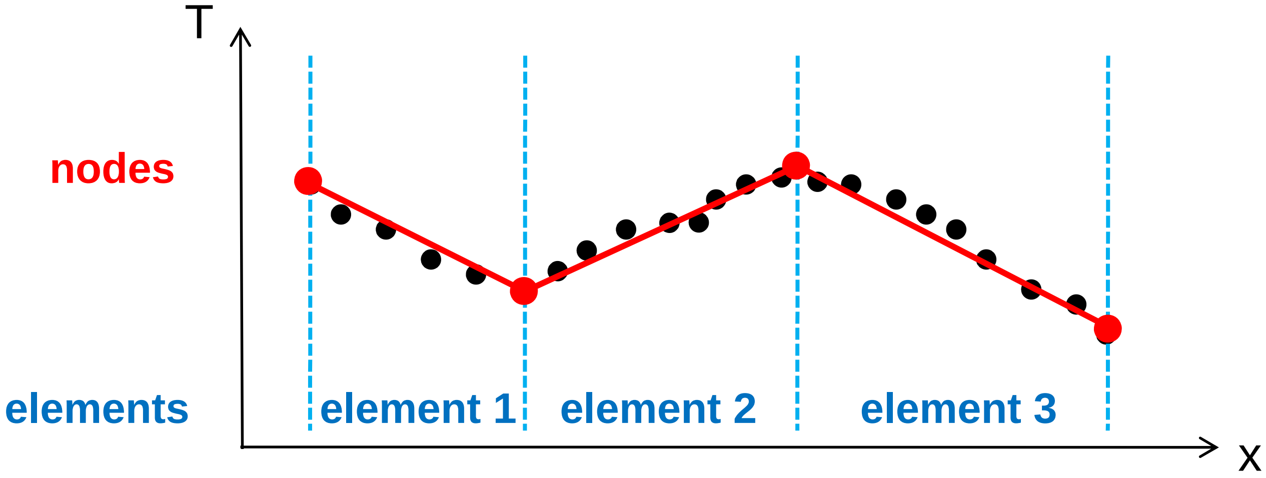

subdomains




Low order interpolation within subdomains

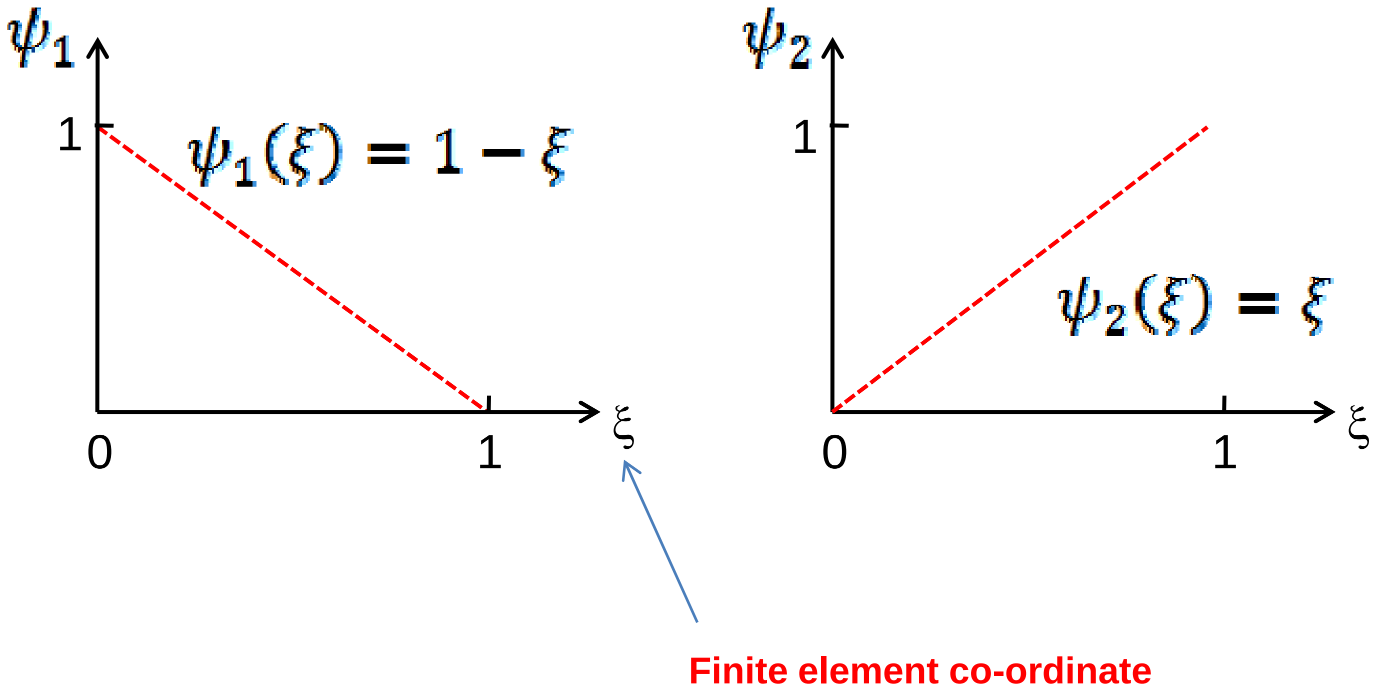
.. but need to ensure continuity across subdomain boundaries.

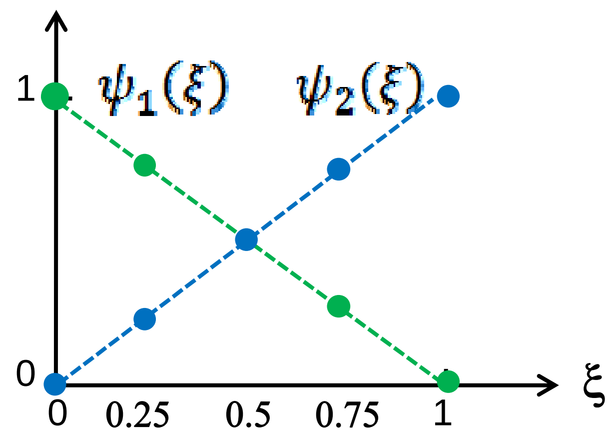
# Nodes & elements

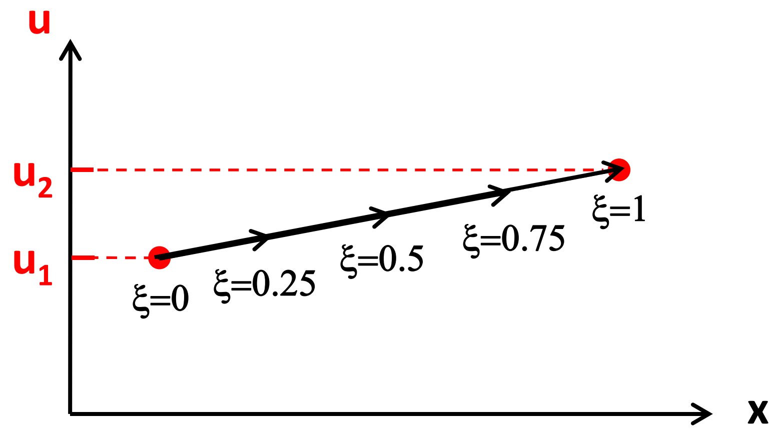



interpolation with basis functions

# Basis functions






$$u(\xi) = \psi_1(\xi)u_1 + \psi_2(\xi)u_2 = (1 - \xi)u_1 + (\xi)u_2$$

$At\ \xi = 0,\quad u(0) = (1 - 0)u_1 + 0u_2 = u_1$

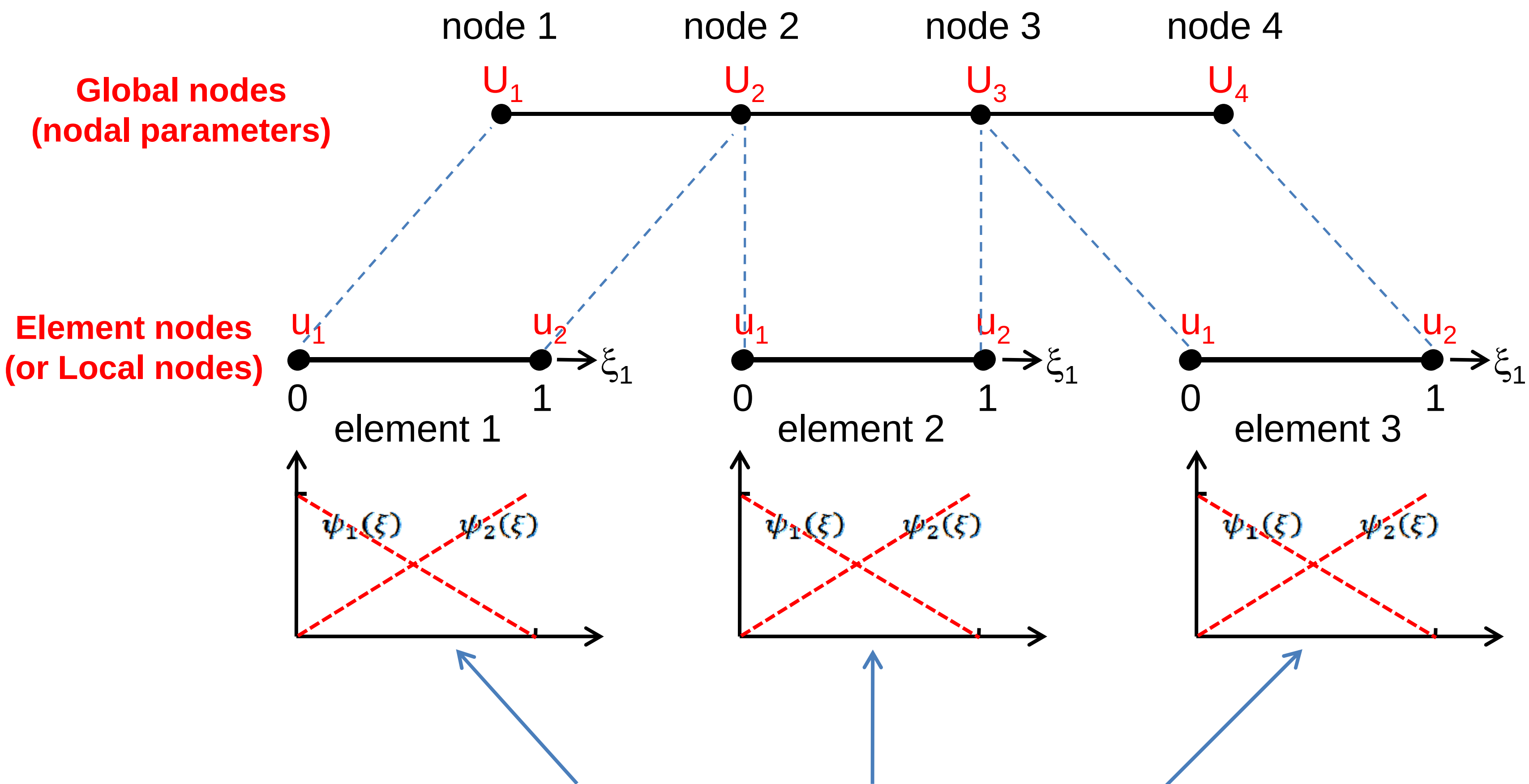
$At\ \xi = 0.25,\quad u(0.25) = (1 - 0.25)u_1 + 0.25u_2 = 0.75u_1 + 0.25u_2$

$At\ \xi = 0.5,\quad u(0.5) = (1 - 0.5)u_1 + 0.5u_2 = 0.5u_1 + 0.5u_2$

$At\ \xi = 0.75,\quad u(0.75) = (1 - 0.75)u_1 + 0.75u_2 = 0.25u_1 + 0.75u_2$

$At\ \xi = 1,\quad u(1) = (1 - 1)u_1 + 1u_2 = u_2$

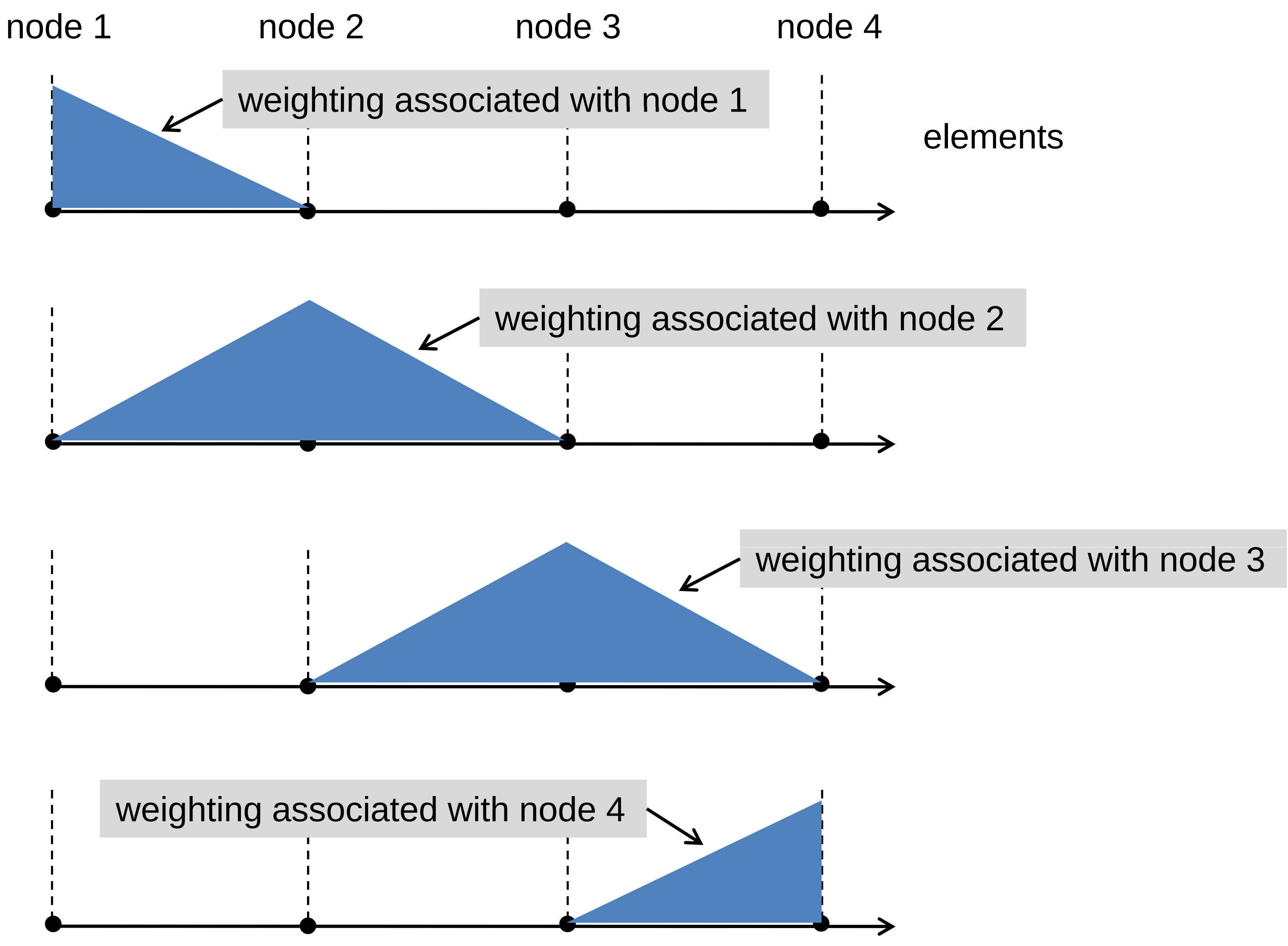
# Connecting elements




$$u(\xi) = \psi_1(\xi)u_1 + \psi_2(\xi)u_2$$

$$= \sum_{n=1,2} \psi_n(\xi)u_n \quad = \psi_n(\xi)u_n \quad (\textit{implied summation})$$

# Influence region for a node

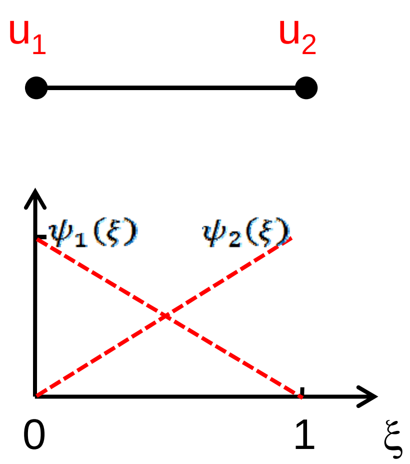


# Properties of basis functions

- Essential property: the basis function associated with a particular node takes the values of:
    - ONE when evaluated at that node
    - ZERO at every other node in the element

- This ensures the linear independence of the basis functions.

- It is also the key to deriving the basis functions from first principles.
  e.g. 1D linear Lagrange basis functions:




$\psi_1(\xi) = a + b\xi$

$\psi_1(\xi) = 1$ at $\xi = 0$ $\rightarrow a = 1$

$\psi_1(\xi) = 0$ at $\xi = 1$ $\rightarrow 1 + b = 0$ $\rightarrow \psi_1(\xi) = 1 - \xi$

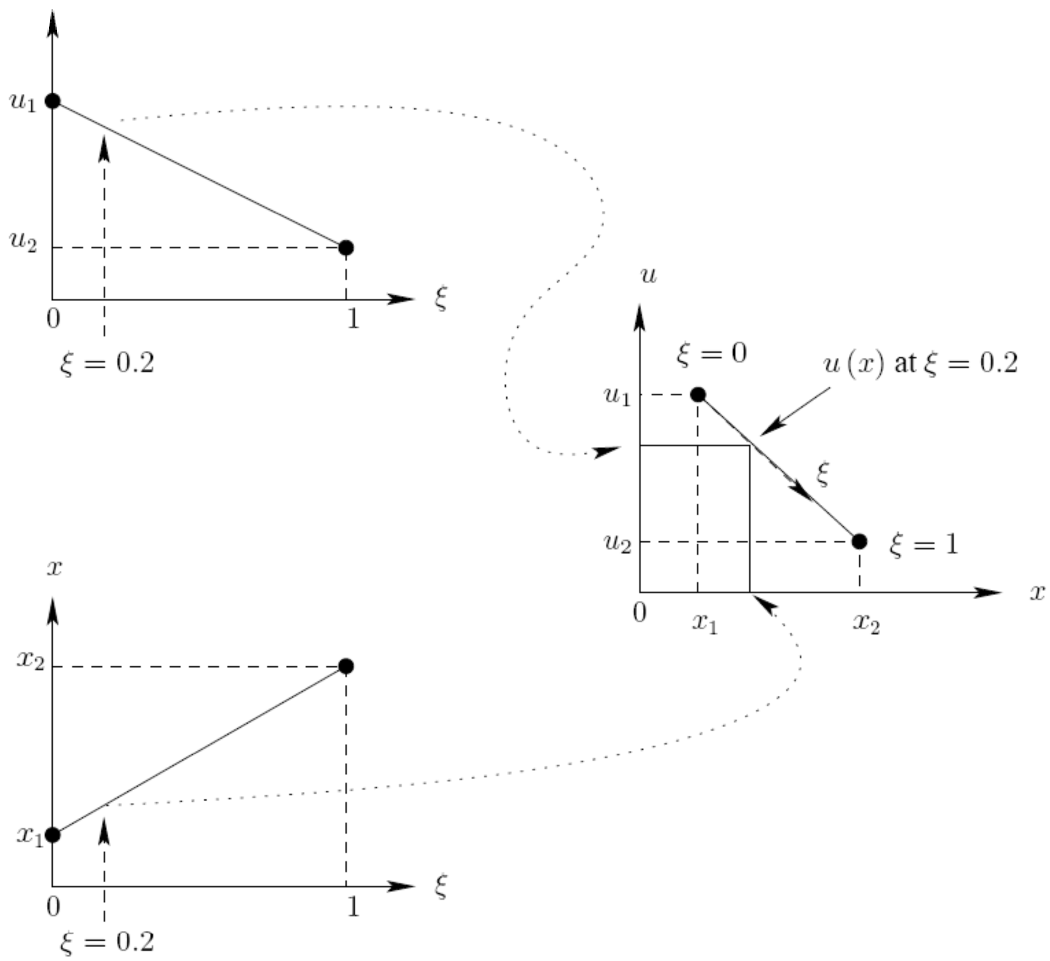
$\psi_2(\xi) = c + d\xi$

$\psi_2(\xi) = 0$ at $\xi = 0$ $\rightarrow c = 0$

$\psi_2(\xi) = 1$ at $\xi = 1$ $\rightarrow d = 1$ $\rightarrow \psi_2(\xi) = \xi$

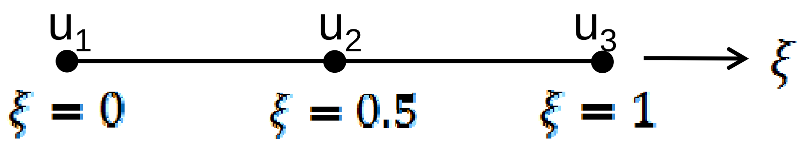
# Interpolating spatial co-ordinates

$$x(\xi) = \psi_n(\xi)x_n \qquad\qquad u(\xi) = \psi_n(\xi)u_n$$

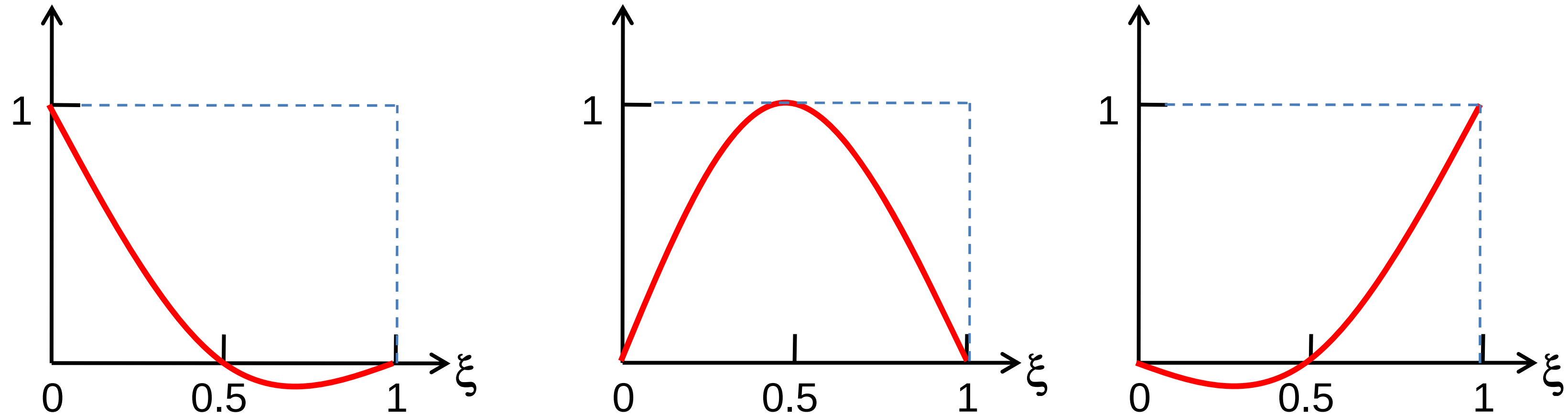


# Quadratic basis functions

3 nodes and 3 basis functions per element



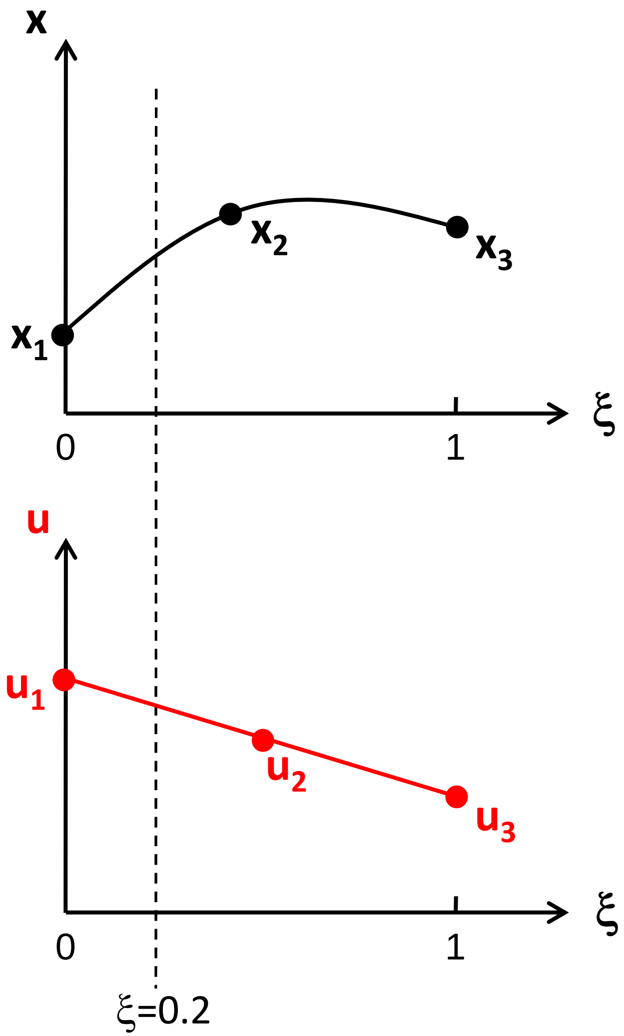

$$u(\xi) = \psi_1(\xi)u_1 + \psi_2(\xi)u_2 + \psi_3(\xi)u_3$$

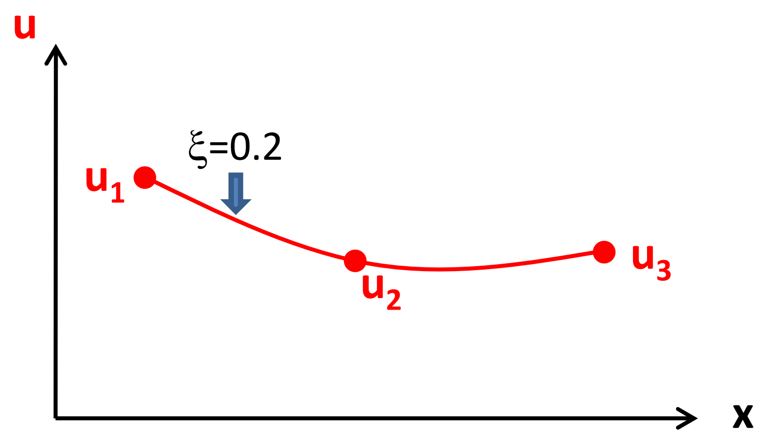



$$\psi_1(\xi) = 2(\xi - 1)(\xi - 0.5) \qquad \psi_2(\xi) = -4\xi(\xi - 1) \qquad \psi_3(\xi) = 2\xi(\xi - 0.5)$$

# u-x-ξ mappings

Consider quadratic element with $x(\xi) = \psi_n(\xi)x_n$ and $u(\xi) = \psi_n(\xi)u_n$



# Bilinear basis functions

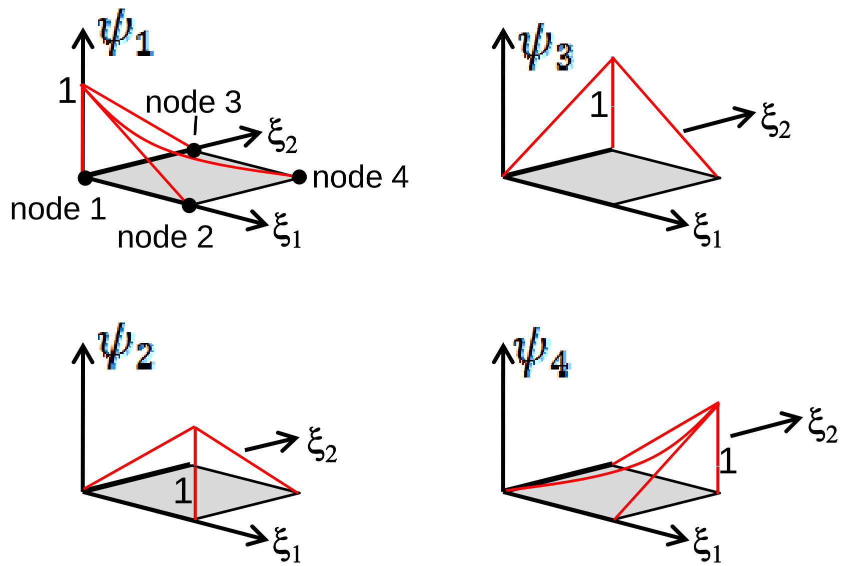
$$u(\xi_1,\xi_2) = \psi_1(\xi_1,\xi_2)u_1 + \psi_2(\xi_1,\xi_2)u_2 + \psi_3(\xi_1,\xi_2)u_3 + \psi_4(\xi_1,\xi_2)u_4 = \sum_{n=1,4} \psi_n(\xi_1,\xi_2)u_n$$

$$\psi_1(\xi_1,\xi_2) = (1-\xi_1)(1-\xi_2)$$

$$\psi_2(\xi_1,\xi_2) = (\xi_1)(1-\xi_2)$$

$$\psi_3(\xi_1,\xi_2) = (1-\xi_1)(\xi_2)$$

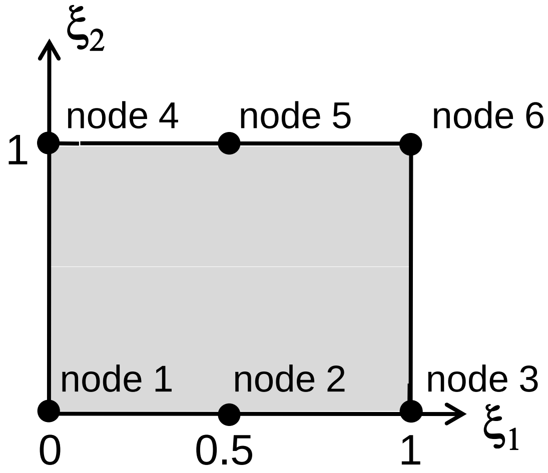
$$\psi_4(\xi_1,\xi_2) = (\xi_1)(\xi_2)$$

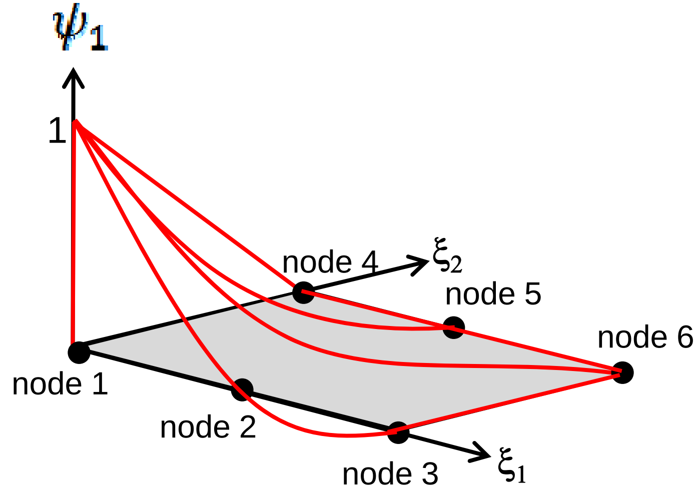



**Note: node numbering important!**

# Quadratic-linear basis functions

$$u(\xi_1, \xi_2) = \sum_{n=1,6} \psi_n(\xi_1, \xi_2) u_n$$



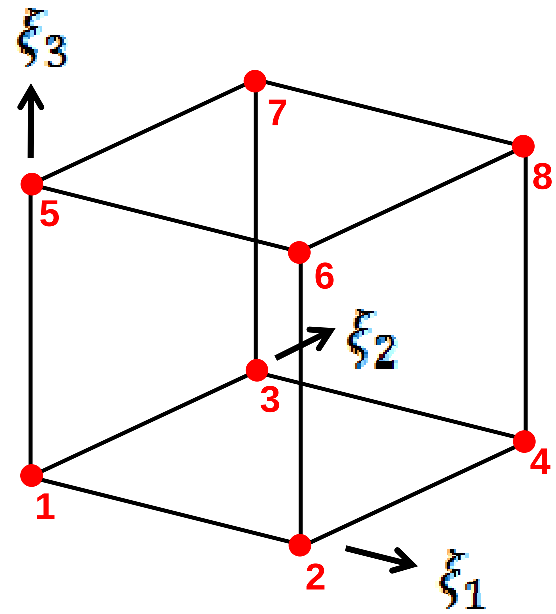

$$\psi_1(\xi_1, \xi_2) = 2(\xi_1 - 1)(\xi_1 - 0.5)(1 - \xi_2)$$

etc

# Trilinear basis functions

$$u(\xi_1, \xi_2, \xi_3) = \sum_{n=1,8} \psi_n(\xi_1, \xi_2, \xi_3) u_n$$

$$\psi_1(\xi_1, \xi_2, \xi_3) = (1 - \xi_1)(1 - \xi_2)(1 - \xi_3)$$
$$\psi_2(\xi_1, \xi_2, \xi_3) = (\xi_1)(1 - \xi_2)(1 - \xi_3)$$
$$\psi_3(\xi_1, \xi_2, \xi_3) = (1 - \xi_1)(\xi_2)(1 - \xi_3)$$
$$\psi_4(\xi_1, \xi_2, \xi_3) = (\xi_1)(\xi_2)(1 - \xi_3)$$
$$\psi_5(\xi_1, \xi_2, \xi_3) = (1 - \xi_1)(1 - \xi_2)(\xi_3)$$
$$\psi_6(\xi_1, \xi_2, \xi_3) = (\xi_1)(1 - \xi_2)(\xi_3)$$
$$\psi_7(\xi_1, \xi_2, \xi_3) = (1 - \xi_1)(\xi_2)(\xi_3)$$
$$\psi_8(\xi_1, \xi_2, \xi_3) = (\xi_1)(\xi_2)(\xi_3)$$



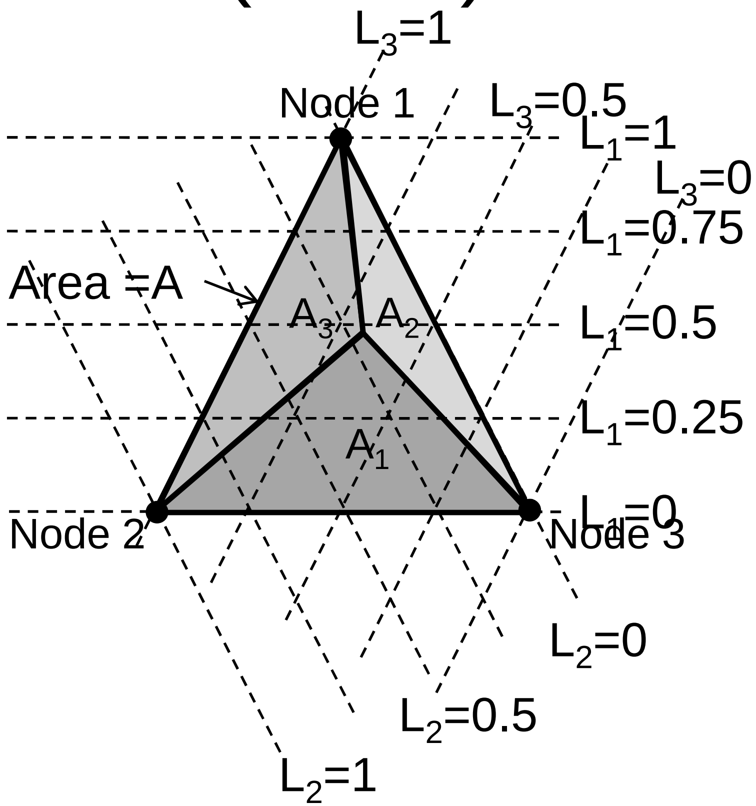
# Simplex basis functions (linear)

$L_1 = A_1 / A$

$L_2 = A_2 / A$

$L_3 = A_3 / A$

$L_1 + L_2 + L_3 = 1$




$$u(L_1, L_2) = \sum_{n=1,3} \psi_n(L_1, L_2) u_n$$

$$\psi_1(L_1, L_2) = L_1$$

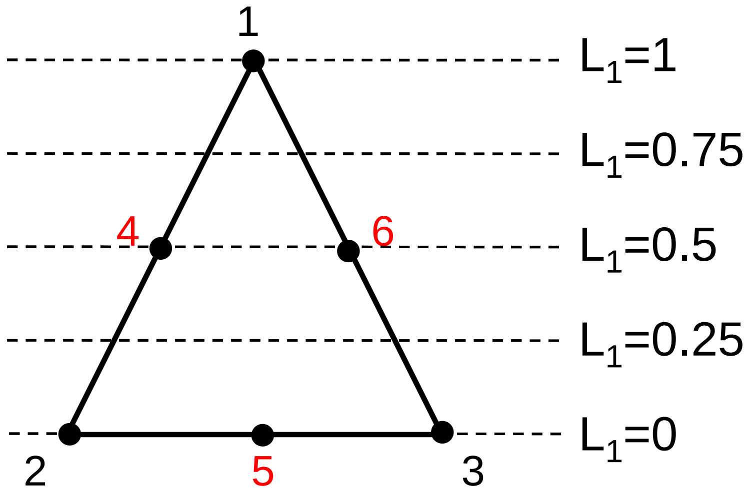
$$\psi_2(L_1, L_2) = L_2$$

$$\psi_3(L_1, L_2) = L_3 = 1 - L_1 - L_2$$

# Simplex basis functions (quadratic)

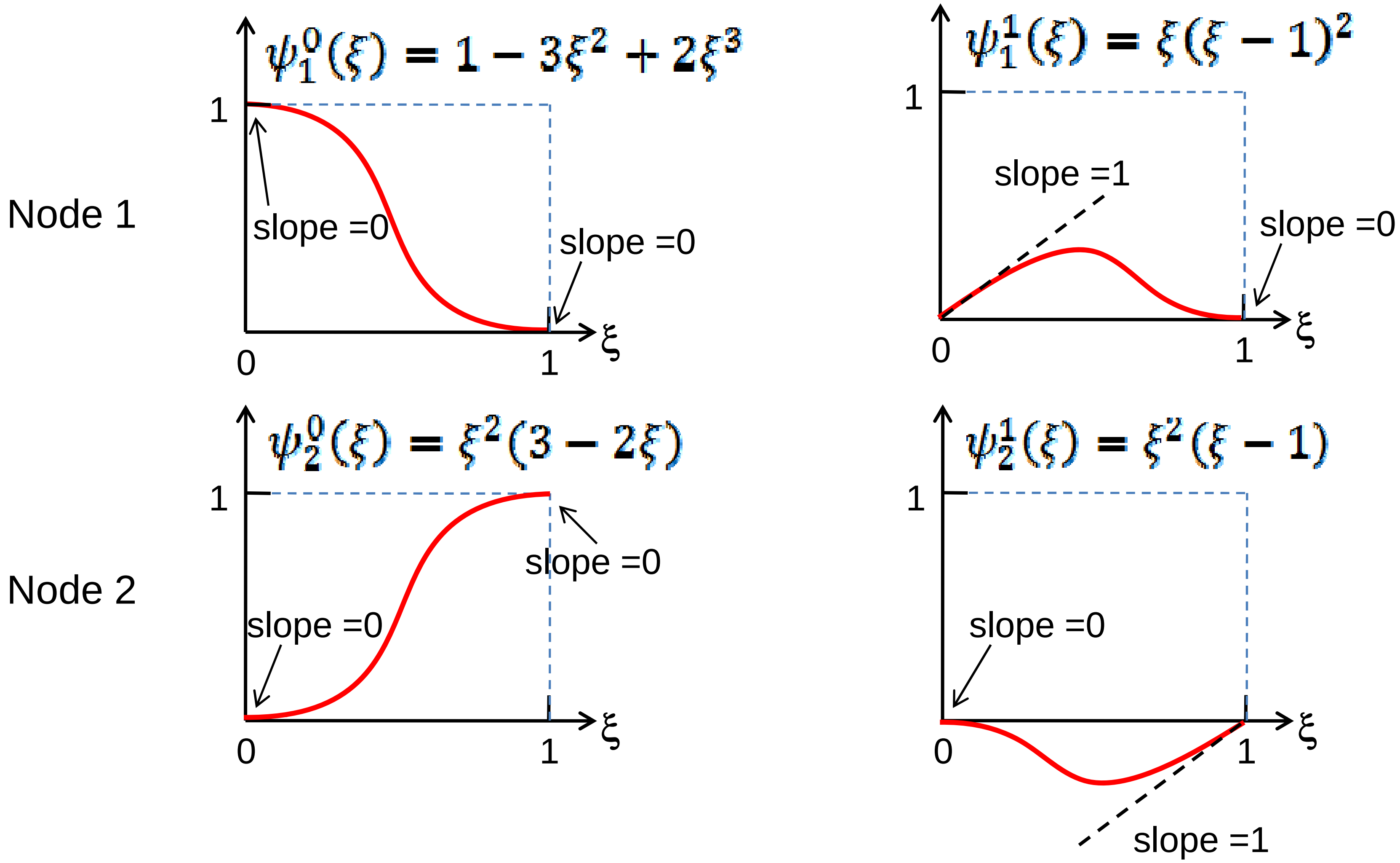
$$u(L_1, L_2) = \sum_{n=1,6} \psi_n(L_1, L_2) u_n$$

$$\psi_1(L_1, L_2) = L_1(2L_1 - 1)$$
$$\psi_2(L_1, L_2) = L_2(2L_2 - 1)$$
$$\psi_3(L_1, L_2) = L_3(2L_3 - 1)$$
$$\psi_4(L_1, L_2) = 4L_1L_2$$
$$\psi_5(L_1, L_2) = 4L_2L_3$$
$$\psi_6(L_1, L_2) = 4L_3L_1$$



# Cubic Hermite basis functions

$$u = \psi_1^0(\xi)u_1 + \psi_1^1(\xi)\left(\frac{du}{d\xi}\right)_1 + \psi_2^0(\xi)u_2 + \psi_2^1(\xi)\left(\frac{du}{d\xi}\right)_2$$



# Hermite basis functions

Scale factors:

store derivatives w.r.t. arc length
need derivatives w.r.t. $\xi$

2D bi-cubic Hermite scalar (**u**): 4 parameters per node
16 parameters per element

2D bi-cubic Hermite geometry in 3D (x,y,z) space:
12 parameters per node
48 parameters per element

# Data fitting

$$S = \sum_t W_t \{x_t - x(\xi_t)\}^2$$

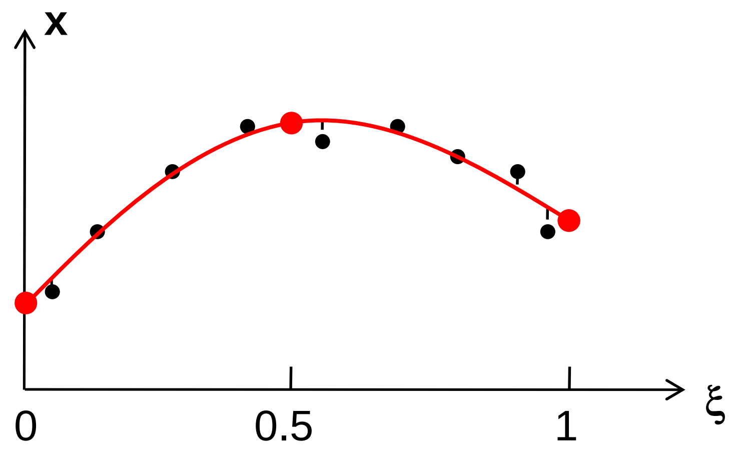
$$x(\xi_t) = \psi_n(\xi_t) x_n$$

$$S = \sum_t W_t \{x_t - \psi_n(\xi_t) x_n\}^2$$

$$For\ m = 1..3 \qquad \frac{\partial S}{\partial x_m} = 0$$

$$2 \sum_t W_t \{x_t - \psi_n(\xi_t) x_n\} \psi_m(\xi_t) = 0$$

$$\left[ \sum_i W_i \psi_m(\xi_i) \psi_n(\xi_i) \right] . x_n = \sum_i W_t \psi_m(\xi_t)\, x_t$$



# Host mesh fitting

Generic mesh

New data set

Choose fiducial points
& calculate residuals

Embed in 'host' mest

Calculate $\xi_t$ points within host mesh

$x(\xi_t) = \psi_n(\xi_t)x_n$ of host

Solve $x_n$ to miminise residuals