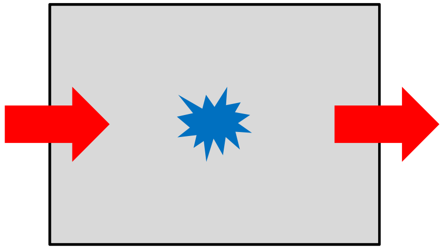
Weighted residuals FEM: steps

1. Form integral equation for PDE
2. Integrate by parts (reduce derivatives on dependent variable)
3. Finite element approximation (basis functions; Galerkin weights)
4. Evaluate element integrals
5. Assemble global matrices
6. Apply boundary conditions
7. Solve global equations
8. (Evaluate RHS terms: e.g. fluxes)

Diffusion equation (e.g. heat equation)



$$\frac{d}{dx}(\textbf{heat flux}) + \textbf{heat sink per unit volume} = 0$$

$$\frac{d}{dx}\left(-k\frac{du}{dx}\right) + q(u,x) = 0$$

e.g. $$\frac{d}{dx}\left(-k\frac{du}{dx}\right) + u = 0$$

Or $R = 0$ where the residual $R = -\frac{d}{dx}\left(k\frac{du}{dx}\right) + u$

With B.C.s: $u(0) = 0$ and $u(1) = 1$ analytic solution is: $u(x) = \frac{e}{e^2 - 1}\left(e^x - e^{-x}\right)$

Weighted residuals (1D diffusion)

$$\int Rw.\,dx = 0$$

e.g. $$\int_0^1 \left\{-\frac{d}{dx}\left(k\frac{du}{dx}\right)w + uw\right\}dx = 0$$

Integration by parts (in 2D or 3D → Green-Gauss theorem)

$$\int_0^1 f\frac{dg}{dx}dx = [f.g]_0^1 - \int_0^1 g\frac{df}{dx}dx$$

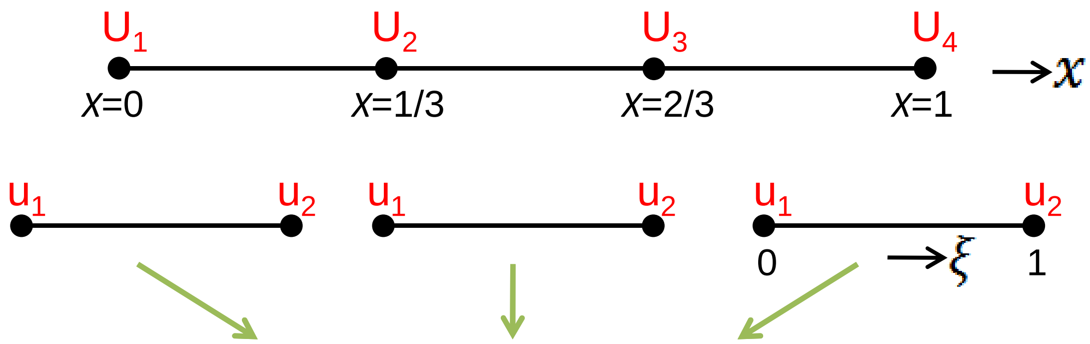
$$\int_0^1 w\frac{d}{dx}(-k\frac{du}{dx})dx = \left[w.(-k\frac{du}{dx})\right]_0^1 - \int_0^1 -k\frac{du}{dx}.\frac{dw}{dx}dx$$

$$\boxed{\int_0^1\left(k\frac{du}{dx}\frac{dw}{dx} + uw\right)dx} = \left[k\frac{du}{dx}w\right]_0^1$$

LHS

Finite element discretisation (1D diffusion)

For 3 element mesh:



$$\int_0^1 .\,dx = \int_0^{\frac{1}{3}} .\,dx + \int_{\frac{1}{3}}^{\frac{2}{3}} .\,dx + \int_{\frac{2}{3}}^{1} .\,dx$$

Change of coordinates:

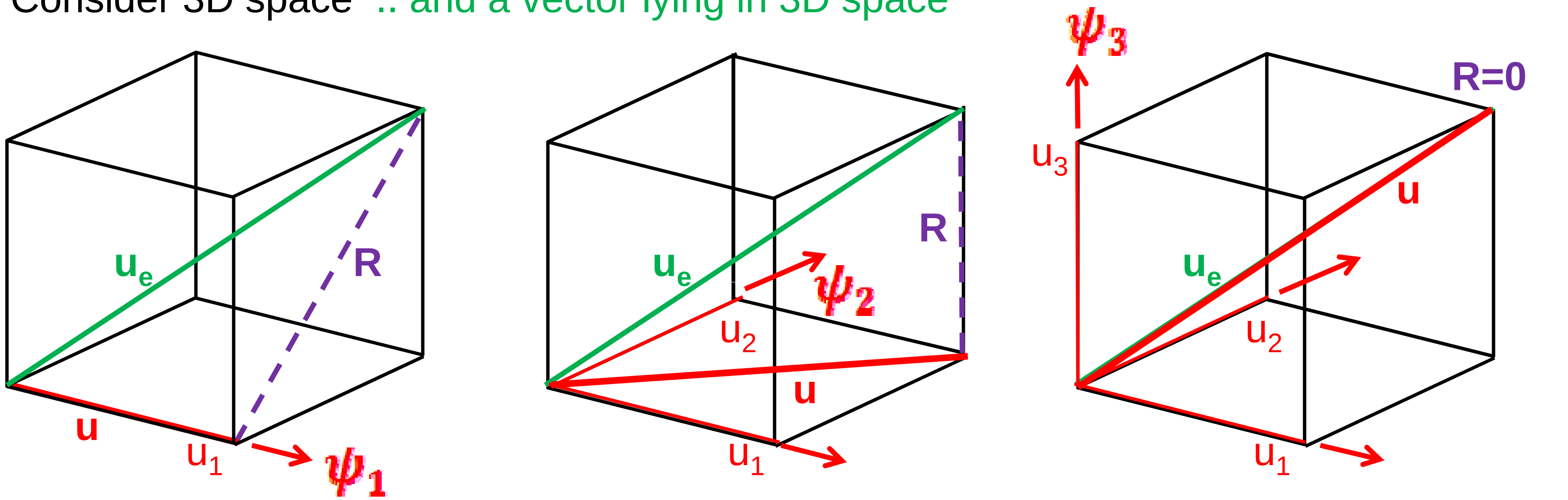
$$\int_{x_1}^{x_2} .\,dx = \int_0^1 .\,J d\xi$$

Where J is Jacobian $J = \left|\frac{\partial x_i}{\partial \xi_j}\right|$

Now LHS = $\int_0^1 \left(k\frac{du}{dx}\frac{dw}{dx} + uw\right) J d\xi$ within each element.

Galerkin approach

Consider 3D space .. and a vector lying in 3D space



$$u = u_1\psi_1 \qquad u = u_1\psi_1 + u_2\psi_2 \qquad u = u_1\psi_1 + u_2\psi_2 + u_3\psi_3$$

Note: **R** is minimum when perpendicular (*orthogonal*) to axes (*space*) that defines **u**.

In function space, orthogonality is $\int Rw.dx = 0$

For *Galerkin* F.E., we choose $w = \psi_1, \psi_2, \psi_3$

Galerkin FEM (1D diffusion)

LHS = $$\int_0^1 \left(k\frac{du}{dx}\frac{dw}{dx} + uw\right) J d\xi$$

$$= u_n . \int_0^1 \left(k\frac{d\psi_n}{d\xi}\frac{d\xi}{dx} . \frac{d\psi_m}{d\xi}\frac{d\xi}{dx} + \psi_n \psi_m\right) J d\xi \qquad = E_{mn} . u_n$$

Element stiffness matrix

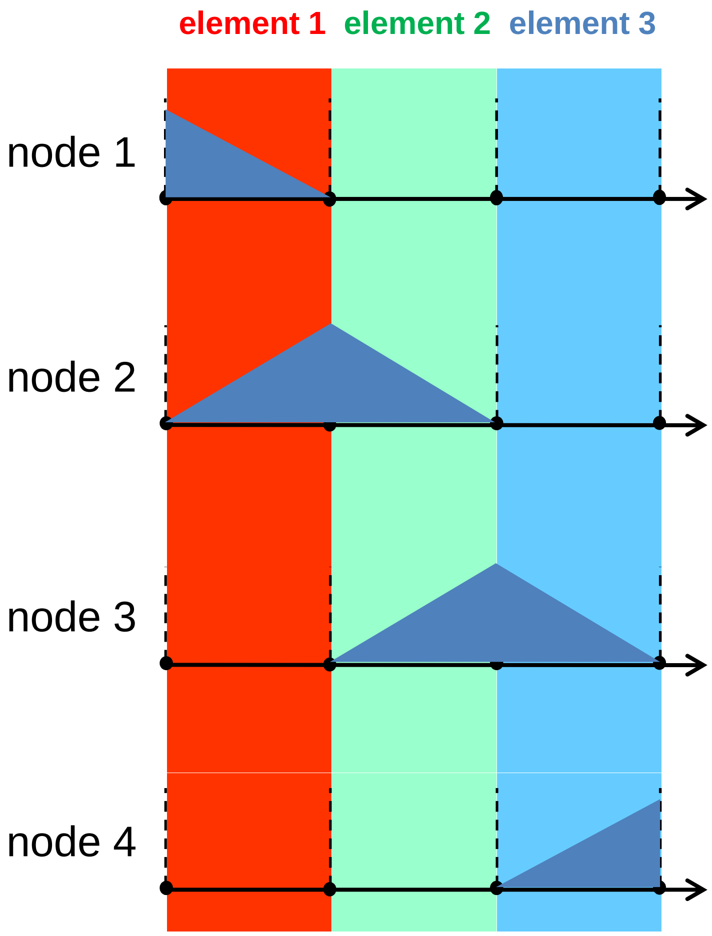
where $$E_{mn} = \int_0^1 \left(k\frac{d\psi_m}{d\xi}\frac{d\xi}{dx} . \frac{d\psi_n}{d\xi}\frac{d\xi}{dx} + \psi_m \psi_n\right) J d\xi$$

$$= \int_0^1 \left(k\frac{d\psi_m}{d\xi} 3 . \frac{d\psi_n}{d\xi} 3 + \psi_m \psi_n\right) \frac{1}{3} d\xi$$

For example $$E_{11} = \int_0^1 \left(k\frac{d\psi_1}{d\xi} 3 . \frac{d\psi_1}{d\xi} 3 + \psi_1 \psi_1\right) \frac{1}{3} d\xi$$

$$= \frac{1}{3}(9k + \frac{1}{3})$$

Global matrix assembly



node 1 2 3 4

$$\begin{bmatrix} X & X & 0 & 0 \\ X & X+X & X & 0 \\ 0 & X & X+X & X \\ 0 & 0 & X & X \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} X \\ X+X=0 \\ X+X=0 \\ X \end{bmatrix}$$

1D Example: Global equations

$$\begin{bmatrix} \frac{28}{9} & -\frac{53}{18} & 0 & 0 \\ -\frac{53}{18} & \frac{28}{9}+\frac{28}{9} & -\frac{53}{18} & 0 \\ 0 & -\frac{53}{18} & \frac{28}{9}+\frac{28}{9} & -\frac{53}{18} \\ 0 & 0 & -\frac{53}{18} & \frac{28}{9} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} -\left(k\frac{du}{dx}\right)\Big|_{x=0} \\ 0 \\ 0 \\ \left(k\frac{du}{dx}\right)\Big|_{x=1} \end{bmatrix}$$

Ku = f

Apply B.C.s: $u(0) = 0$ and $u(1) = 1$

$$\begin{aligned} U_1 &&&& = 0 \\ -\frac{53}{18}U_1 & +\frac{56}{9}U_2 & -\frac{53}{18}U_3 && = 0 \\ & -\frac{53}{18}U_2 & +\frac{56}{9}U_3 & -\frac{53}{18}U_4 & = 0 \\ &&& U_4 & = 1 \end{aligned}$$

Solve: $U_1 = 0$, $U_2 = 0.2885$, $U_3 = 0.6098$, $U_4 = 1$

$$\text{flux entering node 1} = -\left(k\frac{du}{dx}\right)\Big|_{x=0} = -0.8496$$

$$\text{flux entering node 4} = \left(k\frac{du}{dx}\right)\Big|_{x=1} = 1.3157$$

1D Example: solution accuracy

FEM solution: $U_1 = 0$, $U_2 = 0.2885$, $U_3 = 0.6098$, $U_4 = 1$

$$\text{flux entering node } 1 = -\left(k\frac{du}{dx}\right)\bigg|_{x=0} = -0.8496$$

$$\text{flux entering node } 4 = \left(k\frac{du}{dx}\right)\bigg|_{x=1} = 1.3157$$

Analytic solution: $u\left(x\right) = \dfrac{e}{e^2 - 1}\left(e^x - e^{-x}\right)$

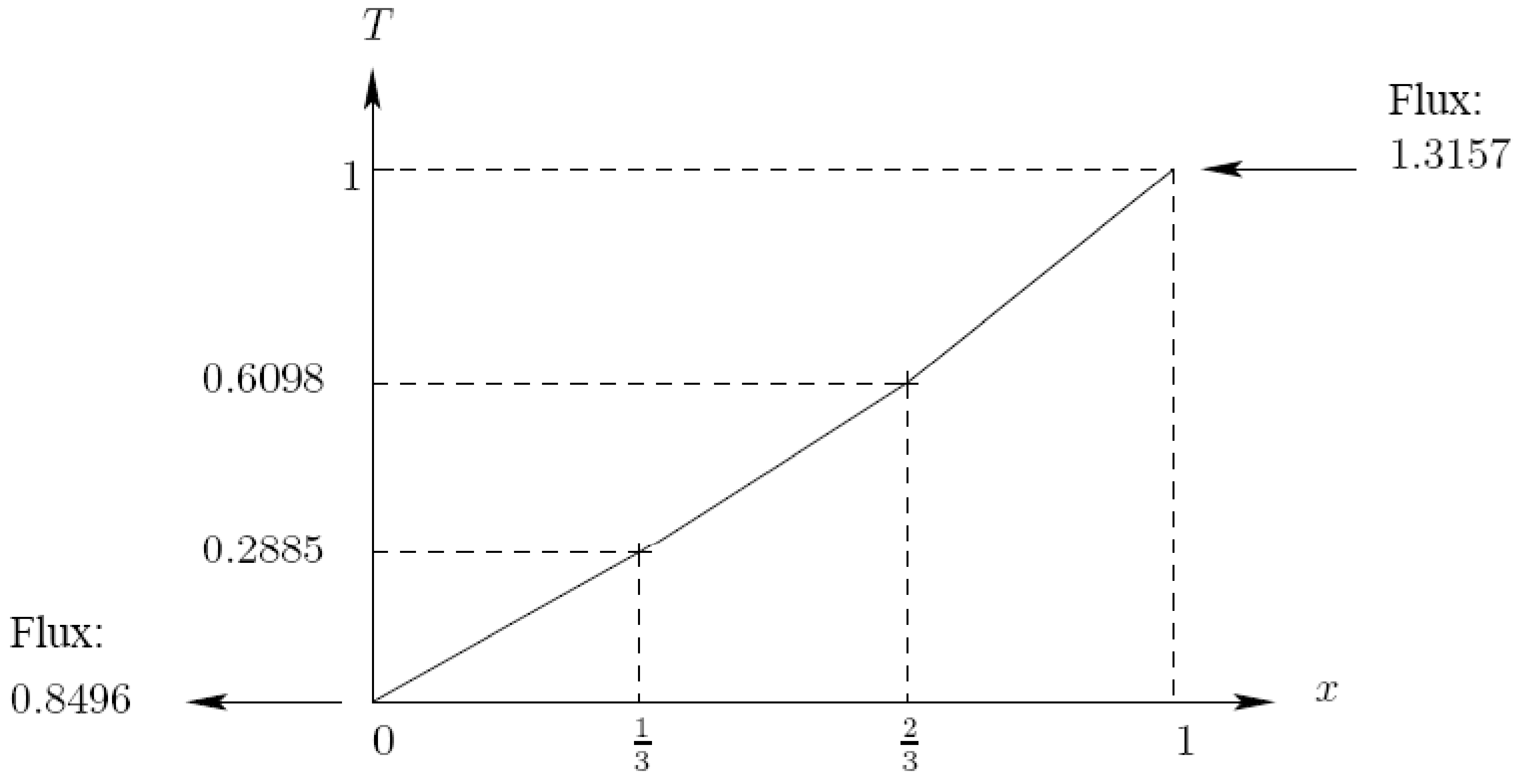


FIGURE 2.2: Finite element solution of one-dimensional heat equation.

3D weighted residuals: example

Steady state heat conduction (diffusion equation): $-\frac{\partial}{\partial x}\left(k_x \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial z}\left(k_z \frac{\partial u}{\partial z}\right) = 0$

Isotropic: $-\nabla \cdot (k \nabla u) = 0$

Weighted residuals: $\int\limits_{\Omega} -\nabla \cdot (k \nabla u)\, \omega \, d\Omega = 0$

Ω is volume domain; Γ is boundary

Integration by parts (Green-Gauss Theorem): $\int\limits_{\Omega} (f \nabla \cdot \nabla g + \nabla f \cdot \nabla g)\, d\Omega = \int\limits_{\Gamma} f \frac{\partial g}{\partial n}\, d\Gamma$

yields: $\int\limits_{\Omega} k \nabla u \cdot \nabla \omega \, d\Omega = \int\limits_{\Gamma} k \frac{\partial u}{\partial n} \omega \, d\Gamma$

where $\nabla u \cdot \nabla \omega = \frac{\partial u}{\partial x_k} \cdot \frac{\partial \omega}{\partial x_k} = \frac{\partial u}{\partial \xi_i} \frac{\partial \xi_i}{\partial x_k} \cdot \frac{\partial \omega}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_k}$ and $\left[\frac{\partial \xi_i}{\partial x_k}\right] = \left[\frac{\partial x_k}{\partial \xi_i}\right]^{-1}$

2D element stiffness matrix: example

Weighted residuals:

$$\int_\Omega k \nabla u \cdot \nabla \omega \, d\Omega = \int_\Gamma k \frac{\partial u}{\partial n} \omega \, d\Gamma$$

Apply FE approximation, and Galerkin assumption:

$$\sum_i u_n \int_\Omega k \left(\frac{\partial \varphi_n}{\partial x}\frac{\partial \varphi_m}{\partial x} + \frac{\partial \varphi_n}{\partial y}\frac{\partial \varphi_m}{\partial y} \right) d\Omega = \int_\Gamma k \frac{\partial u}{\partial n} \varphi_m \, d\Gamma$$

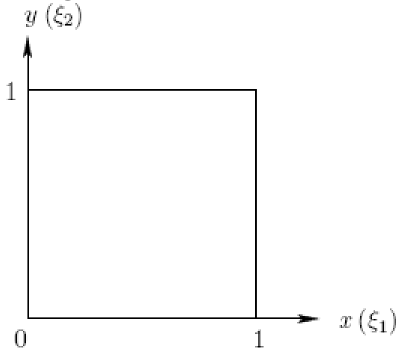
Chain rule:

$$\frac{\partial \varphi_n}{\partial x} = \frac{\partial \varphi_n}{\partial \xi_1}\frac{\partial \xi_1}{\partial x} + \frac{\partial \varphi_n}{\partial \xi_2}\frac{\partial \xi_2}{\partial x} = \frac{\partial \varphi_n}{\partial \xi_i}\frac{\partial \xi_i}{\partial x}$$

$$E_{mn} u_n = F_m$$

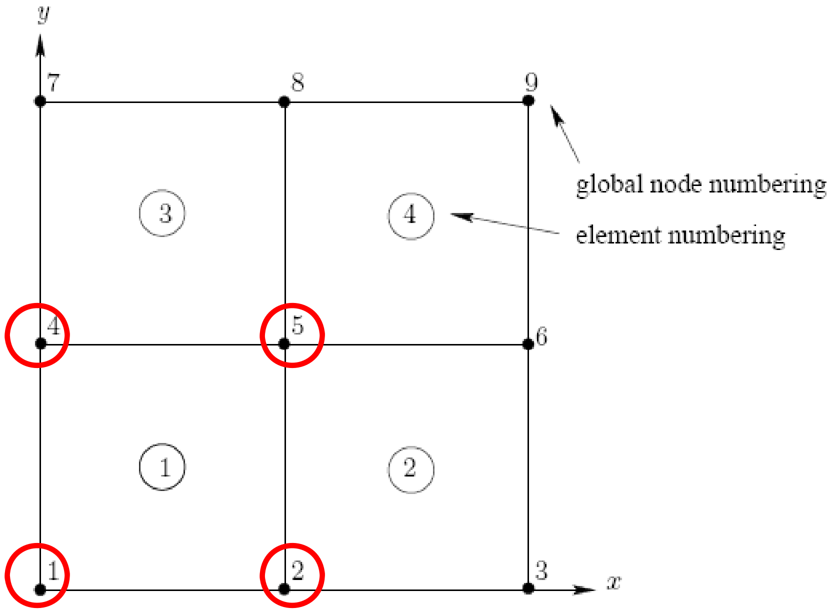
element stiffness matrix

e.g. heat flow in a unit square:



$$k \begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = F_m$$

2D global assembly: example



$$k\begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{6} \\ -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = F_m$$

1 2 3 4 5 6 7 8 9

$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix}\begin{bmatrix} \frac{2}{3} & -\frac{1}{6} & & -\frac{1}{6} & -\frac{1}{3} & & & & \\ -\frac{1}{6} & \frac{2}{3}+\frac{2}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6}-\frac{1}{6} & -\frac{1}{3} & & & \\ & -\frac{1}{6} & \frac{2}{3} & & -\frac{1}{3} & -\frac{1}{6} & & & \\ -\frac{1}{6} & -\frac{1}{3} & & \frac{2}{3}+\frac{2}{3} & -\frac{1}{6}-\frac{1}{6} & & -\frac{1}{6} & -\frac{1}{3} & \\ -\frac{1}{3} & -\frac{1}{6}-\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6}-\frac{1}{6} & \frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3} & -\frac{1}{6}-\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6}-\frac{1}{6} & -\frac{1}{3} \\ & -\frac{1}{3} & -\frac{1}{6} & & -\frac{1}{6}-\frac{1}{6} & \frac{2}{3}+\frac{2}{3} & & -\frac{1}{3} & -\frac{1}{6} \\ & & & -\frac{1}{6} & -\frac{1}{3} & & \frac{2}{3} & -\frac{1}{6} & \\ & & & -\frac{1}{3} & -\frac{1}{6}-\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3}+\frac{2}{3} & -\frac{1}{6} \\ & & & & -\frac{1}{3} & -\frac{1}{6} & & -\frac{1}{6} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \end{bmatrix} = RHS$$

element 1

2D global-to-local mapping: example

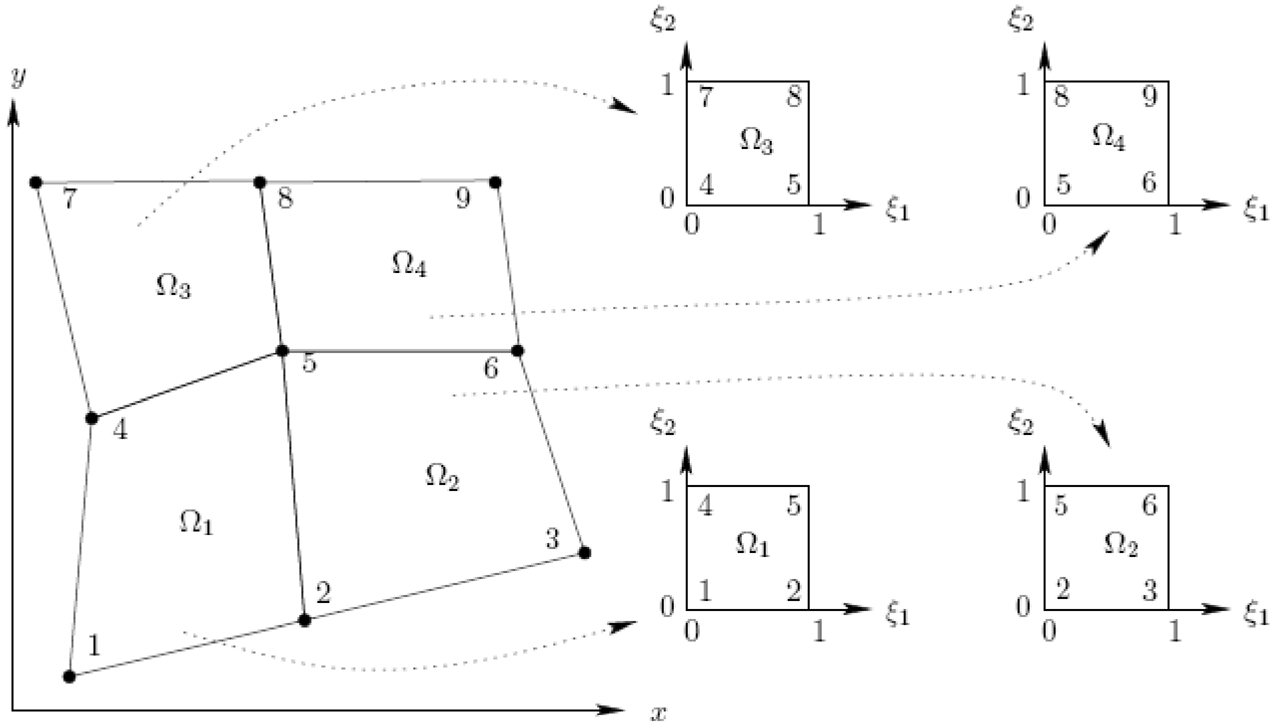


FIGURE 2.5: Mapping each Ω to the ξ_1, ξ_2 plane in a 2×2 element plane.