Finite difference approximations to derivatives

(using Taylor's series expansions)

$$\boxed{\frac{\partial u}{\partial t}} = \boxed{D\frac{\partial^2 u}{\partial x^2}}$$ 1D transient heat equation

Temporal derivative:

$$u_i^{n+1} = u_i^n + \Delta t.\left(\frac{\partial u}{\partial t}\right)_i^n + \mathrm{O}\left(\Delta t^2\right)$$

rearrange:

$$\boxed{\left(\frac{\partial u}{\partial t}\right)_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} + \mathrm{O}\left(\Delta t\right)}$$

Spatial derivative:

$$u_{i+1}^n = u_i^n + \Delta x.\left(\frac{\partial u}{\partial x}\right)_i^n + \frac{1}{2}\Delta x^2.\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{1}{6}\Delta x^3.\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \mathrm{O}\left(\Delta x^4\right)$$

$$u_{i-1}^n = u_i^n - \Delta x.\left(\frac{\partial u}{\partial x}\right)_i^n + \frac{1}{2}\Delta x^2.\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n - \frac{1}{6}\Delta x^3.\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \mathrm{O}\left(\Delta x^4\right)$$

add and rearrange:

$$\boxed{\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + \mathrm{O}\left(\Delta x^2\right)}$$

Finite differences: stability

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$ 1D transient heat equation

$$u_i^{n+1} = u_i^n + D\frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + \mathrm{O}\left(\Delta t^2, \Delta x^2\right)$$

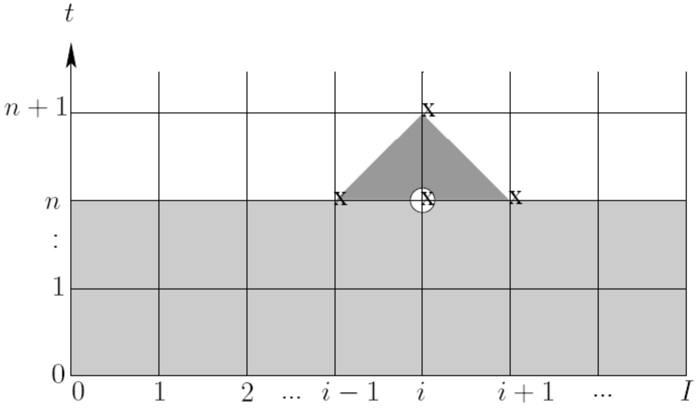
Conditionally stable when:

$$D\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$ *(Courant condition - Von-Neumann stability analysis)*

Finite differences

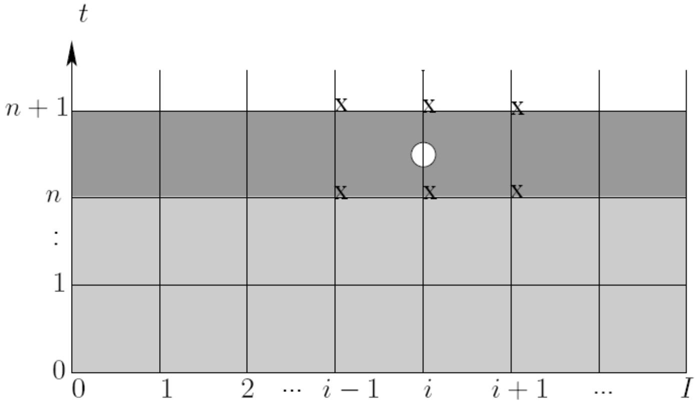
$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$ 1D transient heat equation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = D\left\{\theta\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n+1} + (1-\theta)\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n-}\right\}$$



Explicit forward differences

$$\theta = 0$$



Implicit central differences

$$\theta = \frac{1}{2}$$ *Crank-Nicolson*

Transient Advection-Diffusion FEM

$$\frac{\partial u}{\partial t} + \boldsymbol{v}\cdot\nabla u = D\nabla^2 u + f$$

Weighted residuals: $$\int_\Omega \left(\frac{\partial u}{\partial t} + \boldsymbol{v}\cdot\nabla u - D\nabla^2 u - f\right)\omega \, d\Omega = 0$$

Green-Gauss theorem (integ. by parts):

$$\int_\Omega \left[\left(\frac{\partial u}{\partial t} + \boldsymbol{v}\cdot\nabla u\right)\omega + D\nabla u\cdot\nabla\omega\right] d\Omega = \int_\Omega f\omega \, d\Omega + D\int_{\partial\Omega} \frac{\partial u}{\partial n}\omega \, d\Gamma$$

$$\boldsymbol{M}\frac{d\boldsymbol{u}}{dt} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$$

FE approximation: split into elements and apply $u = \varphi_n u_n$ and $\omega = \varphi_m$

$$M_{mn_e} = \int_0^1 \varphi_m \varphi_n J \, d\xi \qquad K_{mn_e} = \int_0^1 D \frac{\partial \varphi_m}{\partial \xi_i}\frac{\partial \varphi_n}{\partial \xi_j} \cdot \frac{\partial \xi_i}{\partial x_k}\frac{\partial \xi_j}{\partial x_k} J \, d\xi + \int_0^1 v_j \varphi_m \frac{\partial \varphi_n}{\partial \xi_i}\frac{\partial \xi_i}{\partial x_j} J \, d\xi$$

Transient Advection-Diffusion FEM

$$M\frac{d\boldsymbol{u}}{dt} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$$

Temporal integration: finite differences

Spatial integration: finite elements

Finite difference approximation to time derivative (Crank-Nicolson):

$$\boldsymbol{M}\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n}}{\Delta t} + \boldsymbol{K}\left[\theta\boldsymbol{u}^{n+1} + (1-\theta)\,\boldsymbol{u}^{n}\right] = \boldsymbol{F}$$

c.f. 1D finite differences: $\frac{u_i^{n+1} - u_i^{n}}{\Delta t} = D\left\{\theta\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n+1} + (1-\theta)\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{n-}\right\}$

Iterate through time, solving spatial FEM problem (system of equations) at each time step:

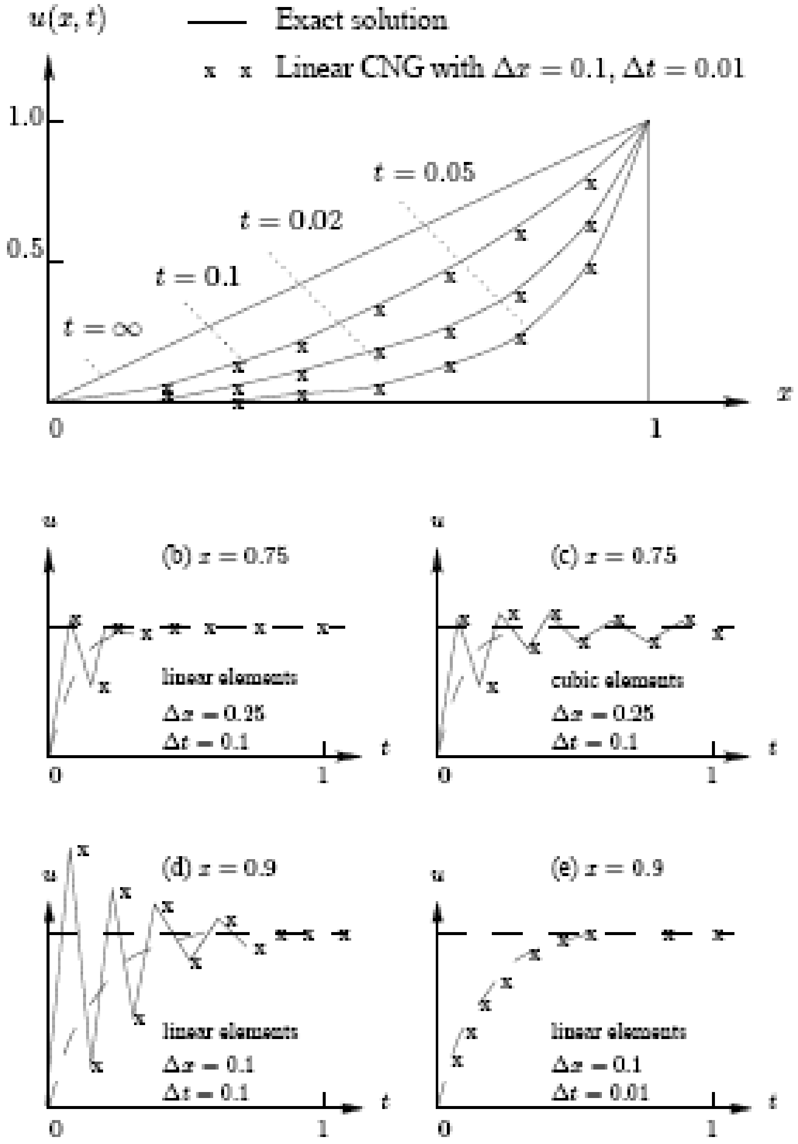
$$\left[\boldsymbol{M} + \theta\Delta t\boldsymbol{K}\right]\boldsymbol{u}^{n+1} = \left[\boldsymbol{M} - (1-\theta)\,\Delta t\boldsymbol{K}\right]\boldsymbol{u}^{n} + \Delta t\ \boldsymbol{F}$$

Transient Advection-Diffusion FEM: stability

Conditionally stable:

$$\Delta t < \frac{1}{(1-\theta)\,\lambda_{\max}}$$

max eigenvalue of $\boldsymbol{M}^{-1}\boldsymbol{K}$.



Mass lumping

$$\boldsymbol{M}\frac{d\boldsymbol{u}}{dt} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}$$

Coupled system of ODEs, because **M** is fully populated.

So for explicit finite differences ($\theta = 0$):
we still need to solve a system of equations at <u>every</u> time step (slow!).

To decouple equations, “lump” row sums of **M** onto diagonals:

$$\boldsymbol{M} = \begin{bmatrix} \frac{1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{36} \\ \frac{1}{18} & \frac{1}{9} & \frac{1}{36} & \frac{1}{18} \\ \frac{1}{18} & \frac{1}{36} & \frac{1}{9} & \frac{1}{18} \\ \frac{1}{36} & \frac{1}{18} & \frac{1}{18} & \frac{1}{9} \end{bmatrix} \xrightarrow{\text{mass lumping}} \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

Mass lumping + explicit finite differences ($\theta = 0$):
<u>NO</u> need to solve system of equations at every time step (fast!).

Mass lumping: phase lag errors