3D Linear Elasticity

Conservation of linear momentum (static equilibrium):

$$\sigma_{ij,j} + b_i = 0 \qquad i,j = 1,2,3 \qquad \sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

Linear elasticity: $\sigma_{ij} = c_{ijkl} e_{kl}$ or for isotropy: $\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij}$

(constitutive relation)

81 material constants

2 material constants

Kinematic relations: $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad i,j = 1,2,3$

Equations: 3 governing eqations + 6 constitutive equations + 6 kinematic relations.

Unknowns: 6 stress components + 6 strain components + 3 displacements

dependent variables

3D Linear Elasticity: weighted residuals

$\sigma_{ij,j} + b_i = \text{R}$

$$\int_\Omega (\sigma_{ij,j} + b_i)\, u_i^*\, d\Omega = 0$$

virtual displacements
(vector weighting field)

Substitute chain rule: $(\sigma_{ij} u_i^*)_{,j} = \sigma_{ij,j} u_i^* + \sigma_{ij} u_{i,j}^*$

divergence theorem: $\int_\Omega \sigma_{ij,j} d\Omega = \int_{\partial\Omega} \sigma_{ij}\, n_j\, d\Gamma$

and Cauchy's formula: $\boldsymbol{t} = \sigma_{ij} n_j \boldsymbol{i}_i$

to get: $$\int_\Omega \sigma_{ij} u_{i,j}^*\, d\Omega = \int_\Omega b_i u_i^*\, d\Omega + \int_{\partial\Omega} t_i u_i^*\, d\Gamma$$

Substitute Principle of Virtual Work:

$$\int_{\partial\Omega} s_i u_i^*\, d\Gamma = \int_{\partial\Omega} t_i u_i^*\, d\Gamma$$

to get: $$\int_\Omega \sigma_{ij} u_{i,j}^*\, d\Omega = \int_\Omega b_i u_i^*\, d\Omega + \int_{\partial\Omega} s_i u_i^*\, d\Gamma$$

3D Linear Elasticity: FEM

$$\int_{\Omega} \sigma_{ij} u^*_{i,j} \, d\Omega = \int_{\Omega} b_i u^*_i \, d\Omega + \int_{\partial\Omega} s_i u^*_i \, d\Gamma$$

Discretise,
apply basis functions,
and Galerkin approach

$$u^*_i = \varphi_m (u^m_i)^*$$

$$u^*_{i,j} = \frac{\partial \varphi_m}{\partial x_j} (u^m_i)^* \qquad = \varphi_{m,k} \frac{\partial \xi_k}{\partial x_j} (u^m_i)^*$$

$$\sum_e \left(\int_{\Omega_e} \sigma_{ij} \varphi_{m,k} \frac{\partial \xi_k}{\partial x_j} \, d\Omega \right) \left(U_i^{\triangle(m,e)} \right)^* = \sum_e \left(\int_{\Omega_e} b_i \varphi_m \, d\Omega + \int_{\partial\Omega_e} s_i \varphi_m \, d\Gamma \right) \left(U_i^{\triangle(m,e)} \right)^*$$

Must hold for any virtual displacement field *U**, thus:

$$\sum_e \int_{\Omega_e} \sigma_{ij} \varphi_{m,k} \frac{\partial \xi_k}{\partial x_j} \, d\Omega = \sum_e \left(\int_{\Omega_e} b_i \varphi_m \, d\Omega + \int_{\partial\Omega_e} s_i \varphi_m \, d\Gamma \right)$$

dependent variables (displacements) in here

RHS terms (e.g. reactions & body forces)

3D Linear Elasticity: FEM (contd.)

$$\sum_e \int_{\Omega_e} \sigma_{ij} \varphi_{m,k} \frac{\partial \xi_k}{\partial x_j} \, d\Omega = \sum_e \left(\int_{\Omega_e} b_i \varphi_m \, d\Omega + \int_{\partial \Omega_e} s_i \varphi_m \, d\Gamma \right)$$

Kinematic relations: $e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad u_j = \varphi_n u_j^n$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(\varphi_n u_i^n\right) + \frac{\partial}{\partial x_i}\left(\varphi_n u_j^n\right)\right) = \frac{1}{2}\left(\frac{\partial \varphi_n}{\partial \xi_l}\frac{\partial \xi_l}{\partial x_j}u_i^n + \frac{\partial \varphi_n}{\partial \xi_l}\frac{\partial \xi_l}{\partial x_i}u_j^n\right)$$

$$e_{kk} = u_{k,k} = \frac{\partial \varphi_n}{\partial \xi_l}\frac{\partial \xi_l}{\partial x_k}u_k^n$$

Linear elasticity:
(constitutive relation)

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

$$\sigma_{ij} = \lambda \delta_{ij}\frac{\partial \varphi_n}{\partial \xi_l}\frac{\partial \xi_l}{\partial x_k}u_k^n + 2\mu\left(\frac{1}{2}\frac{\partial \varphi_n}{\partial \xi_l}\frac{\partial \xi_l}{\partial x_j}u_i^n + \frac{1}{2}\frac{\partial \varphi_n}{\partial \xi_l}\frac{\partial \xi_l}{\partial x_i}u_j^n\right)$$

3D Linear Elasticity: FEM (contd.)

Substitute kinematic relations, constitutive relation, and simplify(!):

$$u_j^n \int\limits_{\Omega_e} \left(\lambda \frac{\partial \xi_l}{\partial x_j} \frac{\partial \xi_k}{\partial x_i} + \mu \left[\frac{\partial \xi_l}{\partial x_i} \frac{\partial \xi_k}{\partial x_j} + \delta_{ij} \frac{\partial \xi_l}{\partial x_p} \frac{\partial \xi_k}{\partial x_p} \right] \right) \frac{\partial \varphi_n}{\partial \xi_l} \frac{\partial \varphi_m}{\partial \xi_k} \, d\Omega = f_{im}$$

displacements (dependent variables)

Element load vector

$$E_{imjn} u_j^n = f_{im}$$

Element stiffness matrix

$$E_{imjn} = \iiint\limits_0^1 \left(\lambda \frac{\partial \xi_l}{\partial x_j} \frac{\partial \xi_k}{\partial x_i} + \mu \left[\frac{\partial \xi_l}{\partial x_i} \frac{\partial \xi_k}{\partial x_j} + \delta_{ij} \frac{\partial \xi_l}{\partial x_p} \frac{\partial \xi_k}{\partial x_p} \right] \right) \frac{\partial \varphi_n}{\partial \xi_l} \frac{\partial \varphi_m}{\partial \xi_k} J(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3$$

$$f_{im} = \iiint\limits_0^1 b_i \varphi_m J(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 + \iint\limits_0^1 s_i \varphi_m J_{2D}(\xi_1, \xi_2) d\xi_1 d\xi_2$$