UPDATING OF NODAL VALUES AND THEIR DERIVATIVES OF THE SLAVE MESH IN HOST MESH FITTING

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1. INTRODUCTION

This document describes the mathematical analysis of the host mesh fitting technique. The slave mesh (the mesh that needs to be transformed/customized) is completely embedded in the host mesh. The nodal parameters of the deformed host mesh are determined by optimising (in this case minimising) the summation of the square of the error (the Euclidean distance) between the landmark and target points. A typical pair of landmark and target point is shown in figures 1.1 and 1.2 together with the un-deformed host/original slave and deformed host/transformed slave meshes. For further details of the fitting of finite element geometric models, the reader is referred to reference 3. Note that all landmark points should be present inside the un-deformed host mesh

Slave mesh could be a line, area or volume mesh (i.e. 1D, 2D or 3D). The host mesh, in general, is a volume mesh. However, for a 1D slave mesh in plane, a 2D host mesh in plane can be used. Here we consider a general case where both slave and host meshes are volume meshes.

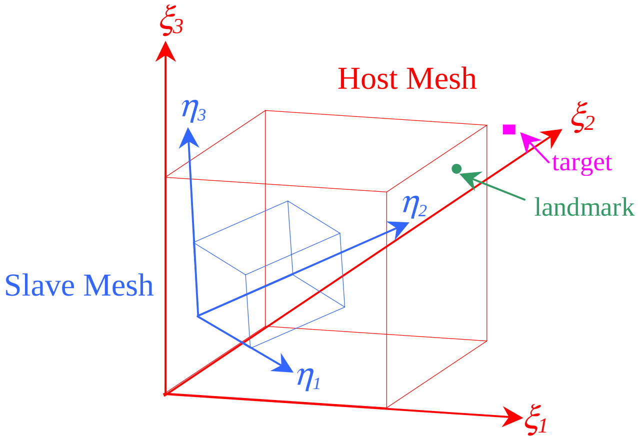

Figure 2.1 Un-deformed Host/Original Slave Mesh

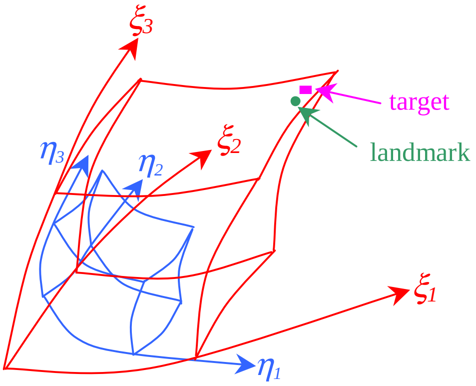

Figure 2.2 Deformed Host/Transformed Slave Mesh

2. UPDATE OF NODAL VALUES

Let us take the local (transformed curvilinear) coordinates of the host mesh as ξ_1, ξ_2 and ξ_3 and those of the slave mesh as η_1, η_2 and η_3.

Since the slave mesh is completely embedded in the host mesh, it is assumed that any deformation in the host mesh will result in deformation in the slave mesh in it. It is also assumed that the relative positions of the nodes of the slave mesh in regard to the host mesh remain unchanged before and after deformation. In other words, local coordinates of the slave nodes with respect to host mesh (i.e. ξ_1, ξ_2 and ξ_3) do not change.

Let's assume the global coordinates and the local coordinates of a slave node in the host element are x,y,z and ξ_1,ξ_2,ξ_3 respectively before deformation.

$$x = \sum_{i=1}^{N} \phi_i(\xi_1,\xi_2,\xi_3) \, . \, X_i \, . \, S_i \tag{2.1}$$

Where N is the number of degrees of freedom (DOFs) of the host element, X_i s are values of the DOFs and the S_i s are the associated scale factors. For instance, a tri-cubic volume element with 8 nodes and no multiple versions has 64 DOFs(variables) per coordinate. These include a nodal value, 3 first derivatives, 3 cross derivatives and a triple derivative per node. [$(1+3+3+1)\times 8 = 64$]. ϕ_i s are the shape or interpolation functions associated with each DOF.

Rearranging equation (2.1),

$$F_x = x - \sum_{i=1}^{N} \phi_i(\xi_1,\xi_2,\xi_3) \, . \, X_i \, . \, S_i = 0 \tag{2.2}$$

Similar expressions can be written for y and z coordinates.

$$F_y = y - \sum_{i=1}^{N} \phi_i(\xi_1,\xi_2,\xi_3) \, . \, Y_i \, . \, S_i = 0 \tag{2.3}$$

$$F_z = z - \sum_{i=1}^{N} \phi_i(\xi_1,\xi_2,\xi_3) \, . \, Z_i \, . \, S_i = 0 \tag{2.4}$$

Equations (2.2),(2.3) and (2.4) constitute to a set non-linear equations and solution to these equations will yield the values of the local coordinates of the slave node embedded in the host mesh. Newton-Raphson method is used to solve these equations as follows.

First start with initial guess of ξ_{1a}, ξ_{2a} and ξ_{3a}. The new values, ξ_{1c}, ξ_{2c} and ξ_{3c} are then calculated from,

$$\begin{bmatrix} \frac{\partial F_x}{\partial \xi_1} & \frac{\partial F_x}{\partial \xi_2} & \frac{\partial F_x}{\partial \xi_3} \\ \frac{\partial F_y}{\partial \xi_1} & \frac{\partial F_y}{\partial \xi_2} & \frac{\partial F_y}{\partial \xi_3} \\ \frac{\partial F_z}{\partial \xi_1} & \frac{\partial F_z}{\partial \xi_2} & \frac{\partial F_z}{\partial \xi_3} \end{bmatrix} . \begin{bmatrix} \xi_{1c} - \xi_{1a} \\ \xi_{2c} - \xi_{2a} \\ \xi_{3c} - \xi_{3a} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{2.5}$$

CMISS uses the following criterion to test the convergence of the iterative solution method.

$$\frac{\left|\sum_{i=1}^{3}\left[(\xi_{ic}-\xi_{ia})^2_{current\ \ iteration}\right]-\sum_{i=1}^{3}\left[(\xi_{ic}-\xi_{ia})^2_{previous\ \ iteration}\right]\right|}{\left|1.0+\sum_{i=1}^{3}(\xi_{ic}-\xi_{ia})^2_{previous\ \ iteration}\right|} < TOLERANCE$$

and the TOLERANCE is determined by,

$$TOLERANCE = DLAMCH(\text{'}eps\text{'}) \times 5.0 = 1.11 \times 10^{-16} \times 5.0 = 5.55 \times 10^{-16}$$

Where DLAMCH is a SCSL (SGI Scientific Computing Software Library) function and with the argument 'eps' it returns a value of $1.11\text{x}10^{-16}$. The factor 5.0 is hard-coded in CMISS for this computation. More details about DLAMCH can be obtained by typing "man DLAMCH" in Unix command prompt. (Note : this tolerance may be too small and some times you may not get the convergence when solving the above equations.)

Once the local coordinates of the slave node are found with respect to host element, they can be substituted back to equations (2.2),(2.3) and (2.4) to get the new nodal values with the DOFs of the deformed host element.

$$\begin{bmatrix} x_{transformed} \\ y_{transformed} \\ z_{transformed} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N}\phi_i(\xi_1,\xi_2,\xi_3).X_{i,DEFORMED}.S_{i,DEFORMED} \\ \sum_{i=1}^{N}\phi_i(\xi_1,\xi_2,\xi_3).X_{i,DEFORMED}.S_{i,DEFORMED} \\ \sum_{i=1}^{N}\phi_i(\xi_1,\xi_2,\xi_3).X_{i,DEFORMED}.S_{i,DEFORMED} \end{bmatrix} \tag{2.6}$$

3. UPDATE OF FIRST DERIVATIVES

Since slave mesh is embedded in the host mesh, we can describe the variation of η_1, η_2 and η_3 using ξ_1, ξ_2 and ξ_3 as follows.

$$\eta_1 = f(\xi_1,\xi_2,\xi_3)$$
$$\eta_2 = f(\xi_1,\xi_2,\xi_3)$$
$$\eta_3 = f(\xi_1,\xi_2,\xi_3)$$

$$\frac{\partial x}{\partial \eta_1} = \frac{\partial x}{\partial \xi_1}.\frac{\partial \xi_1}{\partial \eta_1} + \frac{\partial x}{\partial \xi_2}.\frac{\partial \xi_2}{\partial \eta_1} + \frac{\partial x}{\partial \xi_3}.\frac{\partial \xi_3}{\partial \eta_1} \tag{3.1}$$

$$\frac{\partial y}{\partial \eta_1} = \frac{\partial y}{\partial \xi_1}.\frac{\partial \xi_1}{\partial \eta_1} + \frac{\partial y}{\partial \xi_2}.\frac{\partial \xi_2}{\partial \eta_1} + \frac{\partial y}{\partial \xi_3}.\frac{\partial \xi_3}{\partial \eta_1} \tag{3.2}$$

$$\frac{\partial z}{\partial \eta_1} = \frac{\partial z}{\partial \xi_1}.\frac{\partial \xi_1}{\partial \eta_1} + \frac{\partial z}{\partial \xi_2}.\frac{\partial \xi_2}{\partial \eta_1} + \frac{\partial z}{\partial \xi_3}.\frac{\partial \xi_3}{\partial \eta_1} \tag{3.3}$$

or in matrix form,

$$\begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_3} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} & \frac{\partial y}{\partial \xi_3} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \xi_2} & \frac{\partial z}{\partial \xi_3} \end{bmatrix} . \begin{bmatrix} \frac{\partial \xi_1}{\partial \eta_1} \\ \frac{\partial \xi_2}{\partial \eta_1} \\ \frac{\partial \xi_3}{\partial \eta_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \eta_1} \\ \frac{\partial y}{\partial \eta_1} \\ \frac{\partial z}{\partial \eta_1} \end{bmatrix} \tag{3.4}$$

Note that coefficient matrix is the Jacobian matrix of transformation of the host element at the slave node. The right hand side vector contains the nodal derivatives of the slave mesh with respect to one of its local coordinates, η_1. The unknowns $\frac{\partial \xi_1}{\partial \eta_1}, \frac{\partial \xi_2}{\partial \eta_1}$ and $\frac{\partial \xi_3}{\partial \eta_1}$ remain unchanged for un-deformed host mesh/original slave mesh and deformed host mesh/transformed slave mesh. These unknowns can be determined using the parameters of the un-deformed host element and the old slave nodal parameters

From equation (3.4),

$$\begin{bmatrix} \frac{\partial \xi_1}{\partial \eta_1} \\ \frac{\partial \xi_2}{\partial \eta_1} \\ \frac{\partial \xi_3}{\partial \eta_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_3} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} & \frac{\partial y}{\partial \xi_3} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \xi_2} & \frac{\partial z}{\partial \xi_3} \end{bmatrix}^{-1}_{UN-DEFORMED} \begin{bmatrix} \frac{\partial x}{\partial \eta_1} \\ \frac{\partial y}{\partial \eta_1} \\ \frac{\partial z}{\partial \eta_1} \end{bmatrix}_{ORIGINAL} \tag{3.5}$$

Once unknowns in the LHS vector are found from the above, they can be substituted back to equation (3.4) with the new coefficient (Jacobian) matrix based on the <u>deformed host element</u>. This gives the new nodal derivatives of the slave node.

$$\begin{bmatrix} \frac{\partial x}{\partial \eta_1} \\ \frac{\partial y}{\partial \eta_1} \\ \frac{\partial z}{\partial \eta_1} \end{bmatrix}_{TRANSFORMED} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_3} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} & \frac{\partial y}{\partial \xi_3} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \xi_2} & \frac{\partial z}{\partial \xi_3} \end{bmatrix}_{DEFORMED} . \begin{bmatrix} \frac{\partial \xi_1}{\partial \eta_1} \\ \frac{\partial \xi_2}{\partial \eta_1} \\ \frac{\partial \xi_3}{\partial \eta_1} \end{bmatrix} \quad \text{or}$$

$$\begin{bmatrix} \frac{\partial x}{\partial \eta_1} \\ \frac{\partial y}{\partial \eta_1} \\ \frac{\partial z}{\partial \eta_1} \end{bmatrix}_{TRANSFORMED} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_3} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} & \frac{\partial y}{\partial \xi_3} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \xi_2} & \frac{\partial z}{\partial \xi_3} \end{bmatrix}_{DEFORMED} \times \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_3} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} & \frac{\partial y}{\partial \xi_3} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \xi_2} & \frac{\partial z}{\partial \xi_3} \end{bmatrix}^{-1}_{UN-DEFORMED} \times \begin{bmatrix} \frac{\partial x}{\partial \eta_1} \\ \frac{\partial y}{\partial \eta_1} \\ \frac{\partial z}{\partial \eta_1} \end{bmatrix}_{ORIGINAL} \tag{3.6}$$

Similar expressions can be derived for derivatives with respect to η_2, η_3.

$$\begin{bmatrix} \frac{\partial x}{\partial \eta_2} \\ \frac{\partial y}{\partial \eta_2} \\ \frac{\partial z}{\partial \eta_2} \end{bmatrix}_{TRANS} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_3} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} & \frac{\partial y}{\partial \xi_3} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \xi_2} & \frac{\partial z}{\partial \xi_3} \end{bmatrix}_{DEFORMED} \times \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_3} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} & \frac{\partial y}{\partial \xi_3} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \xi_2} & \frac{\partial z}{\partial \xi_3} \end{bmatrix}_{UN-DEFORMED}^{-1} \times \begin{bmatrix} \frac{\partial x}{\partial \eta_2} \\ \frac{\partial y}{\partial \eta_2} \\ \frac{\partial z}{\partial \eta_2} \end{bmatrix}_{ORIGINAL} \tag{3.7}$$

$$\begin{bmatrix} \frac{\partial x}{\partial \eta_3} \\ \frac{\partial y}{\partial \eta_3} \\ \frac{\partial z}{\partial \eta_3} \end{bmatrix}_{TRANS} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_3} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} & \frac{\partial y}{\partial \xi_3} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \xi_2} & \frac{\partial z}{\partial \xi_3} \end{bmatrix}_{DEFORMED} \times \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_3} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} & \frac{\partial y}{\partial \xi_3} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \xi_2} & \frac{\partial z}{\partial \xi_3} \end{bmatrix}_{UN-DEFORMED}^{-1} \times \begin{bmatrix} \frac{\partial x}{\partial \eta_3} \\ \frac{\partial y}{\partial \eta_3} \\ \frac{\partial z}{\partial \eta_3} \end{bmatrix}_{ORIGINAL} \tag{3.8}$$

4. SECOND ORDER (CROSS) DERIVATIVES

Differentiating equation (3.1) with respect to η_2,

$$\frac{\partial^2 x}{\partial \eta_1 \partial \eta_2} = \frac{\partial x}{\partial \xi_1} . \underline{\frac{\partial^2 \xi_1}{\partial \eta_1 \partial \eta_2}} + \frac{\partial \xi_1}{\partial \eta_1} . \underline{\underline{\frac{\partial^2 x}{\partial \xi_1 \partial \eta_2}}} + \frac{\partial x}{\partial \xi_2} . \underline{\frac{\partial^2 \xi_2}{\partial \eta_1 \partial \eta_2}} + \frac{\partial \xi_2}{\partial \eta_1} . \underline{\underline{\frac{\partial^2 x}{\partial \xi_2 \partial \eta_2}}} + \frac{\partial x}{\partial \xi_3} . \underline{\frac{\partial^2 \xi_3}{\partial \eta_1 \partial \eta_2}} + \frac{\partial \xi_3}{\partial \eta_1} . \underline{\underline{\frac{\partial^2 x}{\partial \xi_3 \partial \eta_2}}} \tag{4.1}$$

The single-line underlined terms are the unknowns we need to solve for. The derivative terms underlined with double lines of the above expression however are not desirable in the current form as they are 'mixed derivatives'. They can further be manipulated in the following manner using the chain rule.

$$\frac{\partial^2 x}{\partial \xi_1 \partial \eta_2} = \frac{\partial}{\partial \eta_2}\left(\frac{\partial x}{\partial \xi_1}\right) = \frac{\partial^2 x}{\partial \xi_1^2} . \frac{\partial \xi_1}{\partial \eta_2} + \frac{\partial^2 x}{\partial \xi_1 \partial \xi_2} . \frac{\partial \xi_2}{\partial \eta_2} + \frac{\partial^2 x}{\partial \xi_1 \partial \xi_3} . \frac{\partial \xi_3}{\partial \eta_2} \tag{4.2}$$

$$\frac{\partial^2 x}{\partial \xi_2 \partial \eta_2} = \frac{\partial}{\partial \eta_2}\left(\frac{\partial x}{\partial \xi_2}\right) = \frac{\partial^2 x}{\partial \xi_1 \partial \xi_2} . \frac{\partial \xi_1}{\partial \eta_2} + \frac{\partial^2 x}{\partial \xi_2^2} . \frac{\partial \xi_2}{\partial \eta_2} + \frac{\partial^2 x}{\partial \xi_2 \partial \xi_3} . \frac{\partial \xi_3}{\partial \eta_2} \tag{4.3}$$

$$\frac{\partial^2 x}{\partial \xi_3 \partial \eta_2} = \frac{\partial}{\partial \eta_2}\left(\frac{\partial x}{\partial \xi_3}\right) = \frac{\partial^2 x}{\partial \xi_1 \partial \xi_3} . \frac{\partial \xi_1}{\partial \eta_2} + \frac{\partial^2 x}{\partial \xi_2 \partial \xi_3} . \frac{\partial \xi_2}{\partial \eta_2} + \frac{\partial^2 x}{\partial \xi_3^2} . \frac{\partial \xi_3}{\partial \eta_2} \tag{4.4}$$

In equations (4.2),(4.3) and (4.4), all terms in the RHS are known. For instance, terms like $\frac{\partial \xi_1}{\partial \eta_2}, \frac{\partial \xi_2}{\partial \eta_2}$ etc. are obtainable from the previous section (first derivatives).

Rearranging equation (4.1),

$$\frac{\partial x}{\partial \xi_1}\cdot\frac{\partial^2 \xi_1}{\partial\eta_1\partial\eta_2}+\frac{\partial x}{\partial \xi_2}\cdot\frac{\partial^2 \xi_2}{\partial\eta_1\partial\eta_2}+\frac{\partial x}{\partial \xi_3}\cdot\frac{\partial^2 \xi_3}{\partial\eta_1\partial\eta_2}=\frac{\partial^2 x}{\partial\eta_1\partial\eta_2}-\frac{\partial\xi_1}{\partial\eta_1}\cdot\frac{\partial^2 x}{\partial\xi_1\partial\eta_2}-\frac{\partial\xi_2}{\partial\eta_1}\cdot\frac{\partial^2 x}{\partial\xi_2\partial\eta_2}-\frac{\partial\xi_3}{\partial\eta_1}\cdot\frac{\partial^2 x}{\partial\xi_3\partial\eta_2}$$

Similar expressions can be derived for the cross derivatives of y and z and the resulting equations can be put into matrix form as follows.

$$\begin{bmatrix}\frac{\partial x}{\partial\xi_1} & \frac{\partial x}{\partial\xi_2} & \frac{\partial x}{\partial\xi_3}\\ \frac{\partial y}{\partial\xi_1} & \frac{\partial y}{\partial\xi_2} & \frac{\partial y}{\partial\xi_3}\\ \frac{\partial z}{\partial\xi_1} & \frac{\partial z}{\partial\xi_2} & \frac{\partial z}{\partial\xi_3}\end{bmatrix}\cdot\begin{bmatrix}\frac{\partial^2\xi_1}{\partial\eta_1\partial\eta_2}\\ \frac{\partial^2\xi_2}{\partial\eta_1\partial\eta_2}\\ \frac{\partial^2\xi_3}{\partial\eta_1\partial\eta_2}\end{bmatrix}=\begin{bmatrix}\frac{\partial^2 x}{\partial\eta_1\partial\eta_2}-\frac{\partial\xi_1}{\partial\eta_1}\cdot\boxed{\frac{\partial^2 x}{\partial\xi_1\partial\eta_2}}-\frac{\partial\xi_2}{\partial\eta_1}\cdot\boxed{\frac{\partial^2 x}{\partial\xi_2\partial\eta_2}}-\frac{\partial\xi_3}{\partial\eta_1}\cdot\boxed{\frac{\partial^2 x}{\partial\xi_3\partial\eta_2}}\\ \frac{\partial^2 y}{\partial\eta_1\partial\eta_2}-\frac{\partial\xi_1}{\partial\eta_1}\cdot\boxed{\frac{\partial^2 y}{\partial\xi_1\partial\eta_2}}-\frac{\partial\xi_2}{\partial\eta_1}\cdot\boxed{\frac{\partial^2 y}{\partial\xi_2\partial\eta_2}}-\frac{\partial\xi_3}{\partial\eta_1}\cdot\boxed{\frac{\partial^2 y}{\partial\xi_3\partial\eta_2}}\\ \frac{\partial^2 z}{\partial\eta_1\partial\eta_2}-\frac{\partial\xi_1}{\partial\eta_1}\cdot\boxed{\frac{\partial^2 z}{\partial\xi_1\partial\eta_2}}-\frac{\partial\xi_2}{\partial\eta_1}\cdot\boxed{\frac{\partial^2 z}{\partial\xi_2\partial\eta_2}}-\frac{\partial\xi_3}{\partial\eta_1}\cdot\boxed{\frac{\partial^2 z}{\partial\xi_3\partial\eta_2}}\end{bmatrix}\tag{4.5}$$

The ‘mixed derivative’ terms in the broken-line boxes must be replaced with suitable expressions such as those given in equations (4.2),(4.3) and (4.4).

Solving the system of linear equations in (4.5) above,

$$\begin{bmatrix}\frac{\partial^2\xi_1}{\partial\eta_1\partial\eta_2}\\ \frac{\partial^2\xi_2}{\partial\eta_1\partial\eta_2}\\ \frac{\partial^2\xi_3}{\partial\eta_1\partial\eta_2}\end{bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial\xi_1} & \frac{\partial x}{\partial\xi_2} & \frac{\partial x}{\partial\xi_3}\\ \frac{\partial y}{\partial\xi_1} & \frac{\partial y}{\partial\xi_2} & \frac{\partial y}{\partial\xi_3}\\ \frac{\partial z}{\partial\xi_1} & \frac{\partial z}{\partial\xi_2} & \frac{\partial z}{\partial\xi_3}\end{bmatrix}^{-1}_{UN-DEF}\times\begin{bmatrix}\frac{\partial^2 x}{\partial\eta_1\partial\eta_2}-\frac{\partial\xi_1}{\partial\eta_1}\cdot\frac{\partial^2 x}{\partial\xi_1\partial\eta_2}-\frac{\partial\xi_2}{\partial\eta_1}\cdot\frac{\partial^2 x}{\partial\xi_2\partial\eta_2}-\frac{\partial\xi_3}{\partial\eta_1}\cdot\frac{\partial^2 x}{\partial\xi_3\partial\eta_2}\\ \frac{\partial^2 y}{\partial\eta_1\partial\eta_2}-\frac{\partial\xi_1}{\partial\eta_1}\cdot\frac{\partial^2 y}{\partial\xi_1\partial\eta_2}-\frac{\partial\xi_2}{\partial\eta_1}\cdot\frac{\partial^2 y}{\partial\xi_2\partial\eta_2}-\frac{\partial\xi_3}{\partial\eta_1}\cdot\frac{\partial^2 y}{\partial\xi_3\partial\eta_2}\\ \frac{\partial^2 z}{\partial\eta_1\partial\eta_2}-\frac{\partial\xi_1}{\partial\eta_1}\cdot\frac{\partial^2 z}{\partial\xi_1\partial\eta_2}-\frac{\partial\xi_2}{\partial\eta_1}\cdot\frac{\partial^2 z}{\partial\xi_2\partial\eta_2}-\frac{\partial\xi_3}{\partial\eta_1}\cdot\frac{\partial^2 z}{\partial\xi_3\partial\eta_2}\end{bmatrix}_{ORIGINAL}$$

or

$$\begin{bmatrix}\frac{\partial^2\xi_1}{\partial\eta_1\partial\eta_2}\\ \frac{\partial^2\xi_2}{\partial\eta_1\partial\eta_2}\\ \frac{\partial^2\xi_3}{\partial\eta_1\partial\eta_2}\end{bmatrix}=\left[\frac{\partial(x,y,z)}{\partial(\xi_1,\xi_2,\xi_3)}\right]^{-1}_{UN-DEF}\times\begin{bmatrix}\frac{\partial^2 x}{\partial\eta_1\partial\eta_2}-\frac{\partial\xi_1}{\partial\eta_1}\cdot\frac{\partial^2 x}{\partial\xi_1\partial\eta_2}-\frac{\partial\xi_2}{\partial\eta_1}\cdot\frac{\partial^2 x}{\partial\xi_2\partial\eta_2}-\frac{\partial\xi_3}{\partial\eta_1}\cdot\frac{\partial^2 x}{\partial\xi_3\partial\eta_2}\\ \frac{\partial^2 y}{\partial\eta_1\partial\eta_2}-\frac{\partial\xi_1}{\partial\eta_1}\cdot\frac{\partial^2 y}{\partial\xi_1\partial\eta_2}-\frac{\partial\xi_2}{\partial\eta_1}\cdot\frac{\partial^2 y}{\partial\xi_2\partial\eta_2}-\frac{\partial\xi_3}{\partial\eta_1}\cdot\frac{\partial^2 y}{\partial\xi_3\partial\eta_2}\\ \frac{\partial^2 z}{\partial\eta_1\partial\eta_2}-\frac{\partial\xi_1}{\partial\eta_1}\cdot\frac{\partial^2 z}{\partial\xi_1\partial\eta_2}-\frac{\partial\xi_2}{\partial\eta_1}\cdot\frac{\partial^2 z}{\partial\xi_2\partial\eta_2}-\frac{\partial\xi_3}{\partial\eta_1}\cdot\frac{\partial^2 z}{\partial\xi_3\partial\eta_2}\end{bmatrix}_{ORIGINAL}\tag{4.6}$$

Where, $\left[\frac{\partial(x,y,z)}{\partial(\xi_1,\xi_2,\xi_3)}\right]$ is the Jacobian matrix of transformation for host element.

For other second-order derivatives,

$$\begin{bmatrix} \dfrac{\partial^2 \xi_1}{\partial \eta_2 \partial \eta_3} \\ \dfrac{\partial^2 \xi_2}{\partial \eta_2 \partial \eta_3} \\ \dfrac{\partial^2 \xi_3}{\partial \eta_2 \partial \eta_3} \end{bmatrix} = \left[\frac{\partial(x,y,z)}{\partial(\xi_1,\xi_2,\xi_3)} \right]_{UN-DEF}^{-1} \times \begin{bmatrix} \dfrac{\partial^2 x}{\partial \eta_2 \partial \eta_3} - \dfrac{\partial \xi_1}{\partial \eta_2} \cdot \dfrac{\partial^2 x}{\partial \xi_1 \partial \eta_3} - \dfrac{\partial \xi_2}{\partial \eta_2} \cdot \dfrac{\partial^2 x}{\partial \xi_2 \partial \eta_3} - \dfrac{\partial \xi_3}{\partial \eta_2} \cdot \dfrac{\partial^2 x}{\partial \xi_3 \partial \eta_3} \\ \dfrac{\partial^2 y}{\partial \eta_2 \partial \eta_3} - \dfrac{\partial \xi_1}{\partial \eta_2} \cdot \dfrac{\partial^2 y}{\partial \xi_1 \partial \eta_3} - \dfrac{\partial \xi_2}{\partial \eta_2} \cdot \dfrac{\partial^2 y}{\partial \xi_2 \partial \eta_3} - \dfrac{\partial \xi_3}{\partial \eta_2} \cdot \dfrac{\partial^2 y}{\partial \xi_3 \partial \eta_3} \\ \dfrac{\partial^2 z}{\partial \eta_2 \partial \eta_3} - \dfrac{\partial \xi_1}{\partial \eta_2} \cdot \dfrac{\partial^2 z}{\partial \xi_1 \partial \eta_3} - \dfrac{\partial \xi_2}{\partial \eta_2} \cdot \dfrac{\partial^2 z}{\partial \xi_2 \partial \eta_3} - \dfrac{\partial \xi_3}{\partial \eta_2} \cdot \dfrac{\partial^2 z}{\partial \xi_3 \partial \eta_3} \end{bmatrix}_{ORIGINAL}$$

(4.7)

$$\begin{bmatrix} \dfrac{\partial^2 \xi_1}{\partial \eta_3 \partial \eta_1} \\ \dfrac{\partial^2 \xi_2}{\partial \eta_3 \partial \eta_1} \\ \dfrac{\partial^2 \xi_3}{\partial \eta_3 \partial \eta_1} \end{bmatrix} = \left[\frac{\partial(x,y,z)}{\partial(\xi_1,\xi_2,\xi_3)} \right]_{UN-DEF}^{-1} \times \begin{bmatrix} \dfrac{\partial^2 x}{\partial \eta_3 \partial \eta_1} - \dfrac{\partial \xi_1}{\partial \eta_3} \cdot \dfrac{\partial^2 x}{\partial \xi_1 \partial \eta_1} - \dfrac{\partial \xi_2}{\partial \eta_3} \cdot \dfrac{\partial^2 x}{\partial \xi_2 \partial \eta_1} - \dfrac{\partial \xi_3}{\partial \eta_3} \cdot \dfrac{\partial^2 x}{\partial \xi_3 \partial \eta_1} \\ \dfrac{\partial^2 y}{\partial \eta_3 \partial \eta_1} - \dfrac{\partial \xi_1}{\partial \eta_3} \cdot \dfrac{\partial^2 y}{\partial \xi_1 \partial \eta_1} - \dfrac{\partial \xi_2}{\partial \eta_3} \cdot \dfrac{\partial^2 y}{\partial \xi_2 \partial \eta_1} - \dfrac{\partial \xi_3}{\partial \eta_3} \cdot \dfrac{\partial^2 y}{\partial \xi_3 \partial \eta_1} \\ \dfrac{\partial^2 z}{\partial \eta_3 \partial \eta_1} - \dfrac{\partial \xi_1}{\partial \eta_3} \cdot \dfrac{\partial^2 z}{\partial \xi_1 \partial \eta_1} - \dfrac{\partial \xi_2}{\partial \eta_3} \cdot \dfrac{\partial^2 z}{\partial \xi_2 \partial \eta_1} - \dfrac{\partial \xi_3}{\partial \eta_3} \cdot \dfrac{\partial^2 z}{\partial \xi_3 \partial \eta_1} \end{bmatrix}_{ORIGINAL}$$

(4.8)

Once the terms in the LHS vectors of equations (4.6), (4.7) and (4.8) are determined, they can be substituted back to equation (4.1) to evaluate the new second order derivatives of the slave node with the Jacobian matrix based on the <u>deformed host element</u> and the <u>original slave nodal parameters</u>.

$$\begin{bmatrix} \dfrac{\partial^2 x}{\partial \eta_1 \partial \eta_2} - \dfrac{\partial \xi_1}{\partial \eta_1} \cdot \dfrac{\partial^2 x}{\partial \xi_1 \partial \eta_2} - \dfrac{\partial \xi_2}{\partial \eta_1} \cdot \dfrac{\partial^2 x}{\partial \xi_2 \partial \eta_2} - \dfrac{\partial \xi_3}{\partial \eta_1} \cdot \dfrac{\partial^2 x}{\partial \xi_3 \partial \eta_2} \\ \dfrac{\partial^2 y}{\partial \eta_1 \partial \eta_2} - \dfrac{\partial \xi_1}{\partial \eta_1} \cdot \dfrac{\partial^2 y}{\partial \xi_1 \partial \eta_2} - \dfrac{\partial \xi_2}{\partial \eta_1} \cdot \dfrac{\partial^2 y}{\partial \xi_2 \partial \eta_2} - \dfrac{\partial \xi_3}{\partial \eta_1} \cdot \dfrac{\partial^2 y}{\partial \xi_3 \partial \eta_2} \\ \dfrac{\partial^2 z}{\partial \eta_1 \partial \eta_2} - \dfrac{\partial \xi_1}{\partial \eta_1} \cdot \dfrac{\partial^2 z}{\partial \xi_1 \partial \eta_2} - \dfrac{\partial \xi_2}{\partial \eta_1} \cdot \dfrac{\partial^2 z}{\partial \xi_2 \partial \eta_2} - \dfrac{\partial \xi_3}{\partial \eta_1} \cdot \dfrac{\partial^2 z}{\partial \xi_3 \partial \eta_2} \end{bmatrix}_{TRANSFORMED} = \left[\frac{\partial(x,y,z)}{\partial(\xi_1,\xi_2,\xi_3)} \right]_{DEF} \times$$

$$\left[\frac{\partial(x,y,z)}{\partial(\xi_1,\xi_2,\xi_3)} \right]_{UN-DEF}^{-1} \times \begin{bmatrix} \dfrac{\partial^2 x}{\partial \eta_1 \partial \eta_2} - \dfrac{\partial \xi_1}{\partial \eta_1} \cdot \dfrac{\partial^2 x}{\partial \xi_1 \partial \eta_2} - \dfrac{\partial \xi_2}{\partial \eta_1} \cdot \dfrac{\partial^2 x}{\partial \xi_2 \partial \eta_2} - \dfrac{\partial \xi_3}{\partial \eta_1} \cdot \dfrac{\partial^2 x}{\partial \xi_3 \partial \eta_2} \\ \dfrac{\partial^2 y}{\partial \eta_1 \partial \eta_2} - \dfrac{\partial \xi_1}{\partial \eta_1} \cdot \dfrac{\partial^2 y}{\partial \xi_1 \partial \eta_2} - \dfrac{\partial \xi_2}{\partial \eta_1} \cdot \dfrac{\partial^2 y}{\partial \xi_2 \partial \eta_2} - \dfrac{\partial \xi_3}{\partial \eta_1} \cdot \dfrac{\partial^2 y}{\partial \xi_3 \partial \eta_2} \\ \dfrac{\partial^2 z}{\partial \eta_1 \partial \eta_2} - \dfrac{\partial \xi_1}{\partial \eta_1} \cdot \dfrac{\partial^2 z}{\partial \xi_1 \partial \eta_2} - \dfrac{\partial \xi_2}{\partial \eta_1} \cdot \dfrac{\partial^2 z}{\partial \xi_2 \partial \eta_2} - \dfrac{\partial \xi_3}{\partial \eta_1} \cdot \dfrac{\partial^2 z}{\partial \xi_3 \partial \eta_2} \end{bmatrix}_{ORIGINAL}$$

(4.9)

5. THIRD ORDER (TRIPLE) DERIVATIVES

The new triple derivative terms like $\frac{\partial^3 x}{\partial \eta_1 \partial \eta_2 \partial \eta_3}$ in the transformed slave mesh can be determined by differentiating the equation (4.1) with respect to η_3 and replacing the mixed derivative terms with appropriate terms and applying the same procedure described in the section 4.

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