TRANSFORMATION OF MESH NODAL VALUES AND DERIVATIVES

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1. INTRODUCTION

Geometric transformation of objects can be put into three categories as follows.

1. Euclidean Transformation
2. Affine Transformation
3. Projective Transformation.

This document discusses first two transformations and their implementation in the finite element meshes.

2. EUCLIDEAN TRANSFORMATION

In Euclidean transformation, only translations and rotations are considered.

2.1 Translation

Let us consider a point in space with its global coordinates being *x,y,z*. After applying translations of Δx, Δy and Δz in the directions of *X,Y* and *Z*, the new coordinates of the point are,

$$x_t = x + \Delta x$$
$$y_t = y + \Delta y$$
$$z_t = z + \Delta z$$

In matrix form,

$$\begin{bmatrix} x_t \\ y_t \\ z_t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{2.1}$$

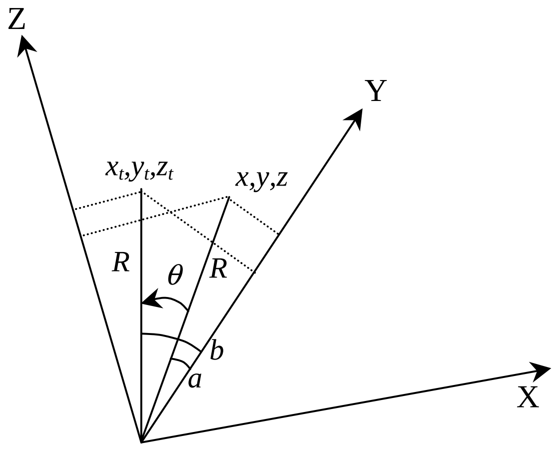
Here we work with a 4 x 4 square matrix so that we can always determine the inverse-transformation.

2.2 Rotation

A rotation could be about X,Y or Z axis. In a right-handed coordinate system, rotation about X and Z axes are similar as the positive direction of angle is in the anti-clock wise direction. However, for the rotation about Y axis, in the same right-handed system, in the anti-clock wise direction the angle measure is negative.

Let us assume the rotation about X, Y and Z are θ, α and β respectively.

Rotation about X axis (only *y* and *z* coordinates change and *x* will remain unchanged),



$$y = R.\cos a \quad : \quad z = R.\sin a \qquad (1)$$

$$y_t = R.\cos b \quad : \quad z_t = R.\sin b \qquad (2)$$

$$\theta = b - a$$

$$y_t = R.\cos(\theta + a) \quad : \quad z_t = R.\sin(\theta + a)$$

$$y_t = R.\cos\theta.\cos a - R.\sin\theta.\sin a \qquad (3)$$

$$z_t = R.\sin\theta.\cos a + R.\cos\theta.\sin a \qquad (4)$$

Substituting (1) & (2) into (3) & (4)

$$y_t = y.\cos\theta - z.\sin\theta$$

$$z_t = y.\sin\theta + z.\sin\theta$$

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

Figure 2.1 Rotation about X axis by θ

$$\begin{bmatrix} x_t \\ y_t \\ z_t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad (2.2)$$

Similarly for rotation about Y axis,

$$\begin{bmatrix} x_t \\ y_t \\ z_t \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad (2.3)$$

and rotation about Z axis is given by,

$$\begin{bmatrix} x_t \\ y_t \\ z_t \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad (2.4)$$

Note that equation (2.3) was obtained by substituting $-\alpha$ for θ in equation (2.2).

3. AFFINE TRANSFORMATION

Affine transformation handles not only translation and rotation but scaling and shearing as well.

3.1 Scaling

Let us assume that the scaling factors in the X,Y and Z directions are S_x, S_y and S_z respectively. The transformation is given by,

$$\begin{bmatrix} x_t \\ y_t \\ z_t \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{3.1}$$

4. COMBINED TRANSLATION, ROTATION AND SCALING

The final position of a transformation that consists of several basic transformations (translations, rotations etc.) depends on the order of the individual processes. For instance, a rotation of 30^0 about Y axis followed by a translation of 50.0 units in the Z direction is different when the transformation is performed in the reverse order. This phenomenon can be explained either algebraically or geometrically. Algebraically, it is due to the simple fact that the commutative law does not hold for the product of two matrices. i.e.

$$[\mathbf{A}] \times [\mathbf{B}] \neq [\mathbf{B}] \times [\mathbf{A}]$$

In CMGUI, a transformation is performed in the following order.

a. Rotation about X
b. Rotation about Y
c. Rotation about Z followed by
d. Translation in X,Y and Z.

Thus, any transformation that is intended to be viewed in the CMGUI must be performed in the above order.

Putting these individual transformations in above order we can determine the overall transformation matrix.

$$\begin{bmatrix} x_t \\ y_t \\ z_t \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{scaling(5)}} . \underbrace{\begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{translation(4)}} . \underbrace{\begin{bmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{rotation about Z(3)}} . \underbrace{\begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{rotation about Y(2)}} . \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{rotation about X(1)}} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{4.1}$$

Here the scaling is the last operation and the rotation about X is the first operation.

Equation (4.1) can be further simplified to give,

$$\begin{bmatrix} x_t \\ y_t \\ z_t \\ 1 \end{bmatrix} = \begin{bmatrix} (\cos\alpha.\cos\beta).S_x & (\sin\theta.\sin\alpha.\cos\beta - \cos\theta.\sin\beta).S_x & (\cos\theta.\sin\alpha.\cos\beta + \sin\theta.\sin\beta).S_x & \Delta x \\ (\cos\alpha.\sin\beta).S_y & (\sin\theta.\sin\alpha.\sin\beta + \cos\theta.\cos\beta).S_y & (\cos\theta.\sin\alpha.\sin\beta - \sin\theta.\cos\beta).S_y & \Delta y \\ (-\sin\alpha).S_z & (\sin\theta.\cos\alpha).S_z & (\cos\theta.\cos\alpha).S_z & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

(4.2)

Equation (4.2) is the overall transformation for rotation about X,Y and Z and translation and scaling along the same axes.

5. TRANSFORMATION OF DERIVATIVES

Differentiating equation (4.2) with respect to ξ_1,

$$\begin{bmatrix} \left(\frac{\partial x}{\partial \xi_1}\right)_t \\ \left(\frac{\partial y}{\partial \xi_1}\right)_t \\ \left(\frac{\partial z}{\partial \xi_1}\right)_t \end{bmatrix} = \begin{bmatrix} (\cos\alpha.\cos\beta).S_x & (\sin\theta.\sin\alpha.\cos\beta - \cos\theta.\sin\beta).S_x & (\cos\theta.\sin\alpha.\cos\beta + \sin\theta.\sin\beta).S_x \\ (\cos\alpha.\sin\beta).S_y & (\sin\theta.\sin\alpha.\sin\beta + \cos\theta.\cos\beta).S_y & (\cos\theta.\sin\alpha.\sin\beta - \sin\theta.\cos\beta).S_y \\ (-\sin\alpha).S_z & (\sin\theta.\cos\alpha).S_z & (\cos\theta.\cos\alpha).S_z \end{bmatrix} . \begin{bmatrix} \frac{\partial x}{\partial \xi_1} \\ \frac{\partial y}{\partial \xi_1} \\ \frac{\partial z}{\partial \xi_1} \end{bmatrix}$$

(5.1)

Since $\Delta x, \Delta y, \Delta z, \theta, \alpha, \beta, S_x, S_y$ and S_z are independent of local coordinate ξ, they are treated as constants in the process of differentiating. Also note that translation terms Δx , Δy and Δz are not present in equation 5.1.

Similarly, new (transformed) derivatives with respect to ξ_2 and higher order derivatives, such as $\frac{\partial^2}{\partial \xi_1 \partial \xi_2}$ are given by,

$$\begin{bmatrix} \left(\dfrac{\partial^2 x}{\partial\xi_1\partial\xi_2}\right)_t \\ \left(\dfrac{\partial^2 y}{\partial\xi_1\partial\xi_2}\right)_t \\ \left(\dfrac{\partial^2 z}{\partial\xi_1\partial\xi_2}\right)_t \end{bmatrix} = \begin{bmatrix} (\cos\alpha.\cos\beta).S_x & (\sin\theta.\sin\alpha.\cos\beta - \cos\theta.\sin\beta).S_x & (\cos\theta.\sin\alpha.\cos\beta + \sin\theta.\sin\beta).S_x \\ (\cos\alpha.\sin\beta).S_y & (\sin\theta.\sin\alpha.\sin\beta + \cos\theta.\cos\beta).S_y & (\cos\theta.\sin\alpha.\sin\beta - \sin\theta.\cos\beta).S_y \\ (-\sin\alpha).S_z & (\sin\theta.\cos\alpha).S_z & (\cos\theta.\cos\alpha).S_z \end{bmatrix} . \begin{bmatrix} \dfrac{\partial^2 x}{\partial\xi_1\partial\xi_2} \\ \dfrac{\partial^2 y}{\partial\xi_1\partial\xi_2} \\ \dfrac{\partial^2 z}{\partial\xi_1\partial\xi_2} \end{bmatrix} \quad (5.2)$$

References.

1. http://www.css.tayloru.edu/~btoll/s99/424/res/mtu/Notes/geometry/geo-tran.htm