

# SMOOTHING OF EDGES AND CORNERS IN TRI-CUBIC VOLUME MESHES

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 Authour : Kumar Mithraratne  
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## 1. INTRODUCTION

Each element of a tri-cubic Hermite volume mesh, if collapsed nodes are not present, possesses 6 faces. These faces can be treated as area (2d) elements and the geometric variation on the faces can therefore be described by using two of the three local coordinates (transformed curvilinear)  $\xi_1$ ,  $\xi_2$  and  $\xi_3$  of the volume element. For instance, the tri-cubic volume element shown in figure 1.1 is made up of six faces and the local coordinates of faces 1 & 2 are  $\xi_2$ - $\xi_3$ , those of faces 3 & 4 are  $\xi_3$ - $\xi_1$  and face 5 & 6 are  $\xi_1$ - $\xi_2$  respectively.

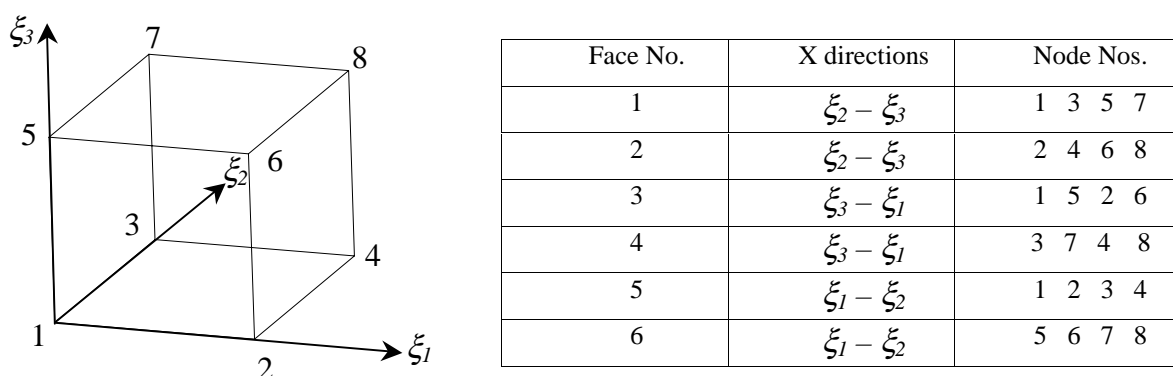


Fig. 1.1 Tri-cubic element      Table 1.1 Faces, their local coordinates and associated nodes

## 2. TRI-CUBIC VOLUME MESHES

A tri-cubic volume mesh, if collapsed nodes are not present, consists of 12 edges (sides) and 8 vortices (corners). Each edge of the mesh could be defined using two  $\xi$  coordinates and each corner using all  $\xi$  coordinates as depicted in figure 2.1

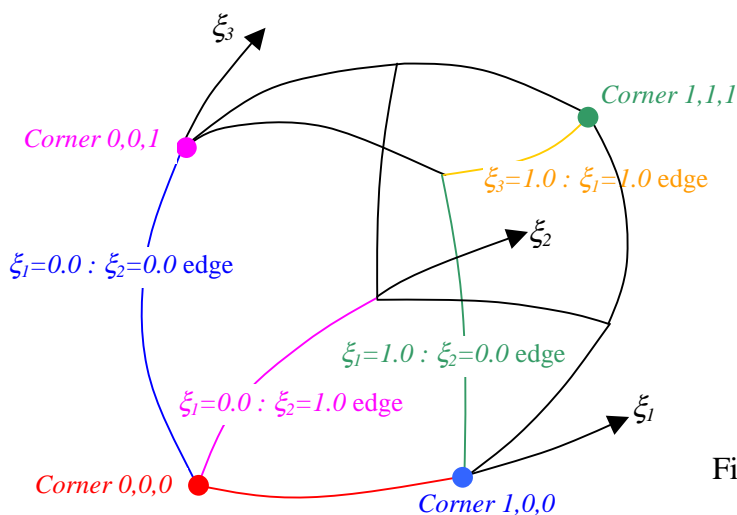


Fig 2.1 Volume mesh with its edges and corners

## 2.1 Derivatives across edges

Let us consider a section parallel to  $\xi_1$ - $\xi_2$  plane through the element shown in figure 1.1.

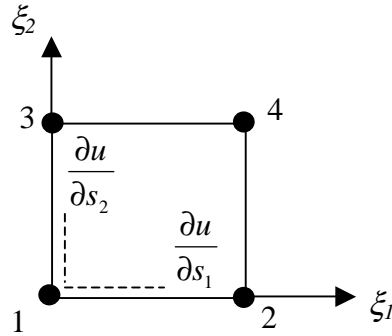


Fig. 2.2 A section parallel to  $\xi_1$ - $\xi_2$  plane.

As can be seen from fig. 2.2, the edge,  $\xi_1=0.0 : \xi_2=0.0$  going through node 1. The derivatives at the node 1 are given by,

$$\left(\frac{\partial u}{\partial s_1}\right)_1 = \frac{u_2 - u_1}{S_{12}} \quad \text{and} \quad \left(\frac{\partial u}{\partial s_2}\right)_1 = \frac{u_3 - u_1}{S_{13}} \quad (2.1)$$

Note that at any given node, a derivative is defined as,

$$\frac{\partial u}{\partial s} = \frac{u_{\xi=1.0} - u_{\xi=0.0}}{\Delta s} \quad (2.2)$$

It is clear from equation (2.1) that at the node 1,  $\left(\frac{\partial u}{\partial s_1}\right)_1 \neq \left(\frac{\partial u}{\partial s_2}\right)_1$ . This results in discontinuity of the derivative across the node 1 when moving along 2-1-3 (or 3-2-1). We can however define the derivatives at the node 1 conforming to the same definition in equation (2.2) but considering the arc 2-1-3 instead of arc 2-1 as shown in figure 2.3.

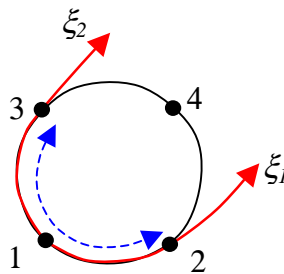


Fig 2.3 Alternative formulation of derivatives at node 1.

$$\left(\frac{\partial u}{\partial s_1}\right)_1 = \frac{u_{\xi_1=1.0} - u_{\xi_1=0.0}}{\Delta s} = \frac{u_2 - u_3}{S_{12} + S_{13}} \quad (2.3)$$

Note that at node 3,  $\xi_1$  is still equal to 0.0.

Similarly for the derivative in the  $\xi_2$  direction,

$$\left(\frac{\partial u}{\partial s_2}\right)_1 = \frac{u_{\xi_2=1.0} - u_{\xi_2=0.0}}{\Delta s} = \frac{u_3 - u_2}{S_{12} + S_{13}} \quad (2.4)$$

Derivatives specified by equations (2.3) and (2.4) guarantee the smooth arc across node 1 on the  $\xi_1$ - $\xi_2$  plane.

Thus the condition necessary to satisfy to obtain a smooth edge at the node 1 is,

$$\left(\frac{\partial u}{\partial s_1}\right)_1 + \left(\frac{\partial u}{\partial s_2}\right)_1 = 0 \quad (2.5)$$

It can easily be shown from a similar analysis that the constraint required to be maintained at the node 4 is identical to the equation (2.5).

$$\left(\frac{\partial u}{\partial s_1}\right)_4 + \left(\frac{\partial u}{\partial s_2}\right)_4 = 0 \quad (2.6)$$

As far as the nodes 2 and 3 are concerned, however, the constraints that need to be stipulated are slightly different.

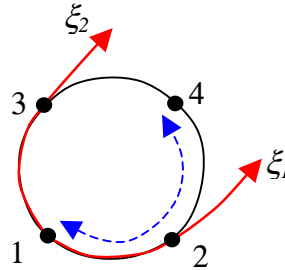


Fig 2.4 Derivative formulation at node 2

For derivative in  $\xi_1$  direction,

$$\left(\frac{\partial u}{\partial s_1}\right)_2 = \frac{u_{\xi_1=1.0} - u_{\xi_1=0.0}}{\Delta s} = \frac{u_4 - u_1}{S_{12} + S_{24}} \quad (2.7)$$

$$\left(\frac{\partial u}{\partial s_2}\right)_2 = \frac{u_{\xi_2=1.0} - u_{\xi_2=0.0}}{\Delta s} = \frac{u_4 - u_1}{S_{12} + S_{24}} \quad (2.8)$$

The above equations result in,

$$\left(\frac{\partial u}{\partial s_1}\right)_2 - \left(\frac{\partial u}{\partial s_2}\right)_2 = 0 \quad (2.9)$$

Similarly at node 3,

$$\left(\frac{\partial u}{\partial s_1}\right)_3 - \left(\frac{\partial u}{\partial s_2}\right)_3 = 0 \quad (2.10)$$

A similar approach could be used for other edges of the volume mesh. Table 2.1 summarizes the constraints required to be imposed to obtain a volume mesh with smooth edges.

Edge (side)	Constrain
$\xi_I=0.0$ and $\xi_2=0.0$	$\frac{\partial u}{\partial s_1} + \frac{\partial u}{\partial s_2} = 0$
$\xi_I=1.0$ and $\xi_2=0.0$	$\frac{\partial u}{\partial s_1} - \frac{\partial u}{\partial s_2} = 0$
$\xi_I=0.0$ and $\xi_2=1.0$	$\frac{\partial u}{\partial s_1} - \frac{\partial u}{\partial s_2} = 0$
$\xi_I=1.0$ and $\xi_2=1.0$	$\frac{\partial u}{\partial s_1} + \frac{\partial u}{\partial s_2} = 0$
$\xi_2=0.0$ and $\xi_3=0.0$	$\frac{\partial u}{\partial s_2} + \frac{\partial u}{\partial s_3} = 0$
$\xi_2=1.0$ and $\xi_3=0.0$	$\frac{\partial u}{\partial s_2} - \frac{\partial u}{\partial s_3} = 0$
$\xi_2=0.0$ and $\xi_3=1.0$	$\frac{\partial u}{\partial s_2} - \frac{\partial u}{\partial s_3} = 0$
$\xi_2=1.0$ and $\xi_3=1.0$	$\frac{\partial u}{\partial s_2} + \frac{\partial u}{\partial s_3} = 0$
$\xi_3=0.0$ and $\xi_I=0.0$	$\frac{\partial u}{\partial s_3} + \frac{\partial u}{\partial s_1} = 0$
$\xi_3=1.0$ and $\xi_I=0.0$	$\frac{\partial u}{\partial s_3} - \frac{\partial u}{\partial s_1} = 0$
$\xi_3=0.0$ and $\xi_I=1.0$	$\frac{\partial u}{\partial s_3} - \frac{\partial u}{\partial s_1} = 0$
$\xi_3=1.0$ and $\xi_I=1.0$	$\frac{\partial u}{\partial s_3} + \frac{\partial u}{\partial s_1} = 0$

Table 2.1 Summary of constraints to be imposed for smooth edges.

## **2.2 Implementation of smoothing constraints with Lagrange multipliers**

Although these constraints are element based, it can be shown that they are global constraints as well. For example, in ‘face fitting’ of volume meshes, we determine the face parameters that minimize the face objective function (the summation of the square of data error) and then assemble to formulate the mesh or global parameters. Once the global stiffness or coefficient matrix and global right hand side vector are assembled, they can be modified by introducing the coefficients associated with Lagrange multipliers for each constraints arising from edge nodes.